SYMMETRIC MATROID POLYTOPES AND THEIR GENERATION

JÜRGEN BOKOWSKI, DAVID BREMNER, AND GÁBOR GÉVAY

A . Matroid polytopes form an intermediate structure useful in searching for realizable convex spheres. In this article we present a class of self-polar 3-spheres that motivated research in the inductive generation of matroid polytopes, along with two new methods of generation.

1. I

The study of polyhedra within the framework of oriented matroids has become a natural approach. Methods for enumerating combinatorial types of convex polytopes inductively within the Euclidean setting alone have not been established. In contrast, the oriented matroid concept allows one to generate matroid polytopes inductively. Matroid polytopes, when not interesting in their own right as topological balls with certain sphere properties, form an intermediate structure to search for realizable convex spheres. We provide in this article an interesting class of self-polar 3-spheres that stimulated research in this area. What are effective methods of generating matroid polytopes with prescribed properties? Having in mind open problems for which a corresponding solution is still open, we present the class of 3-spheres of Gábor Gévay that were found independently by other authors as well. We discuss two new algorithmical methods of David Bremner and of Jürgen Bokowski for generating matroid polytopes that were tested in this context.

2. S - 3-

Here we describe an infinite series of self-polar polyhedral 3-spheres which were found first by the third author [22], and later, independently, by others [36, 38].

2.1. **Description of the structure.** We use two regular n-gons with vertex sets $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ lying in two completely orthogonal linear 2-subspaces and both located on the unit 3-sphere $\mathbb{S}^3$. We

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denote by a^i, a_i and b^i_j the midpoints of the line segments u_iu_{i+1}, v_iv_{i+1} and u_iv_j, respectively. Note that throughout this section, all indices are taken modulo n.

We define convex 3-cells as follows:

- For any a^i, $i \in \{1,\dots,n\}$ and any a^j, $j \in \{1,\dots,n\}$, we define the convex hull

$$P^i_j := \operatorname{conv}\{a^i, b^i_j, b^{i+1}_j, b^{i+1}_{j+1}, b^i_{j+1}, a_j\}.$$

- For any pair (a^{i-1}, a^i), $i \in \{1,\dots,n\}$, we define the pyramids

$$(P^i)_1 := \operatorname{conv}\{a^{i-1}, b^i_1, b^i_2, \dots, b^i_n\},$$

$$(P^i)_2 := \operatorname{conv}\{b^i_1, b^i_2, \dots, b^i_n, a^i\}.$$

- For any pair (a_{i-1}, a_i), $i \in \{1,\dots,n\}$, we define the pyramids

$$(P_i)_1 := \operatorname{conv}\{a_{i-1}, b^1_i, b^2_i, \dots, b^n_i\},$$

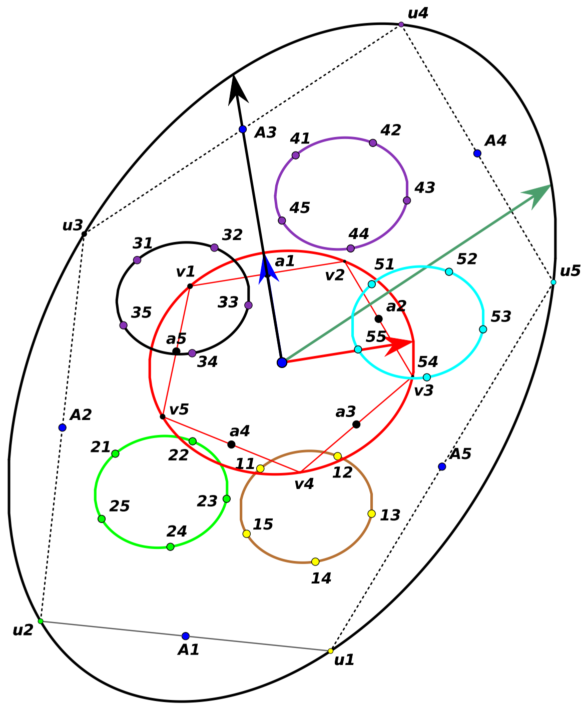
$$(P_i)_2 := \operatorname{conv}\{b^1_i, b^2_i, \dots, b^n_i, a_i\}.$$

In addition we define the polyhedra $P^i := (P^i)_1 \cup (P^i)_2$ and $P_i := (P_i)_1 \cup (P_i)_2$.

The convex hull P^i_j is a 3-polytope which forms (combinatorially) an octahedron since its vertices are the midpoints of the 6 edges of the tetrahedron $T(i,i+1,j,j+1) := \operatorname{conv}\{u_i, u_{i+1}, v_j, v_{j+1}\}$. The set of all tetrahedra T form the boundary of the *free sum* $\operatorname{conv}(U \cup V)$ of $\operatorname{conv}U$ and $\operatorname{conv}V$.

The interiors $\operatorname{int}P^i_j$, $\operatorname{int}(P^k)_p$, and $\operatorname{int}(P_l)_q$, $k,l \in \{1,\dots,n\}$, $p,q \in \{1,2\}$, are pairwise disjoint. E.g. an arbitrary interior point in a pyramid $(P^k)_p$ can be written as a convex combination with at least three non-zero coefficients for the points b^k_s, $s \in \{1,\dots,n\}$, whereas an interior point of P^i_j cannot have such a representation. All the 3-dimensional cells P^i_j, P^k and P_l form together a polyhedral 3-sphere with altogether $(n+2)n$ facets. We denote this sphere by GS_n.

In Figures 1 to 4 we have depicted a planar affine projection from 4-space in the case $n = 5$. The projection shows, apart from all vertices, in particular two octahedra, P^1_5, P^2_5 (Figure 2) and two unions of pyramids $(P^1)_1 \cup (P^1)_2$ and $(P_1)_1 \cup (P_1)_2$ (Figure 3). For each octahedron four non-adjacent subfacets belong to other octahedra, while the other subfacets belong to unions of pyramids (see Figure 4). All 2-faces of a union P_i or P^i are 2-faces of octahedra. When we project a cell P_i, or P^i, radially from the center onto the boundary of the free sum $\operatorname{conv}(U \cup V)$, we see this image as a union of $2n$ tetrahedra around a vertex of $\operatorname{conv}(U \cup V)$.



F 1. Planar Affine Projection for $n = 5$.

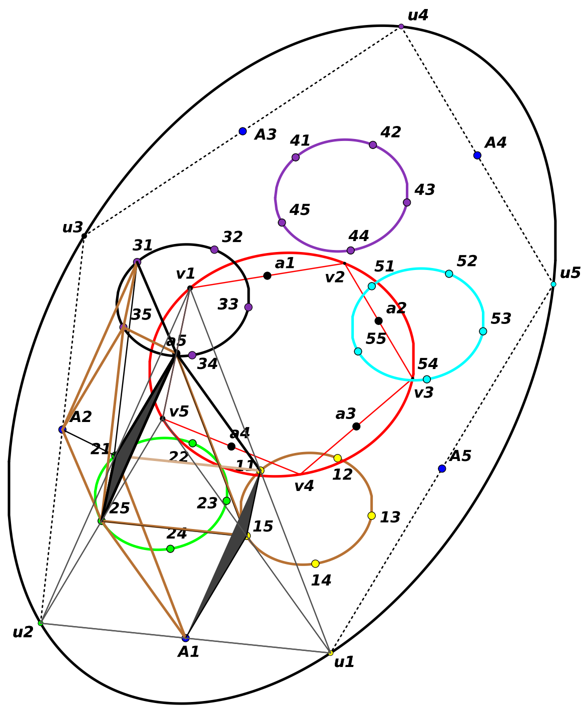
2.2. **Symmetry properties.** In what follows we make distinction between the *combinatorial symmetry group* and the *geometric symmetry group* of a structure under investigation and we use the notation Aut(.) and Sym(.), respectively. In general, the former, being the group of combinatorial automorphisms, may be larger in the sense that it contains a proper subgroup isomorphic to the latter, which is the group of (Euclidean) isometries leaving the structure invariant.

Just as the starting point for describing the structure of GS_n was the set $U \cup V$, here we establish first the symmetry properties of conv($U \cup V$). This is a 4-polytope which we shall denote by P_{nn}. We describe its symmetry properties in terms of Coxeter groups.

Since the symmetry group of a regular n-gon is D_n, the dihedral group of order $2n$, the symmetry group of P_{nn} obviously contains the direct product $D_n \times D_n$ as a subgroup. The whole symmetry group Sym(P_{nn}) is an extension of this direct product by a transformation of order 2 that interchanges U and V.

In Coxeter's notation, we have the following relation, see [15], p. 563:

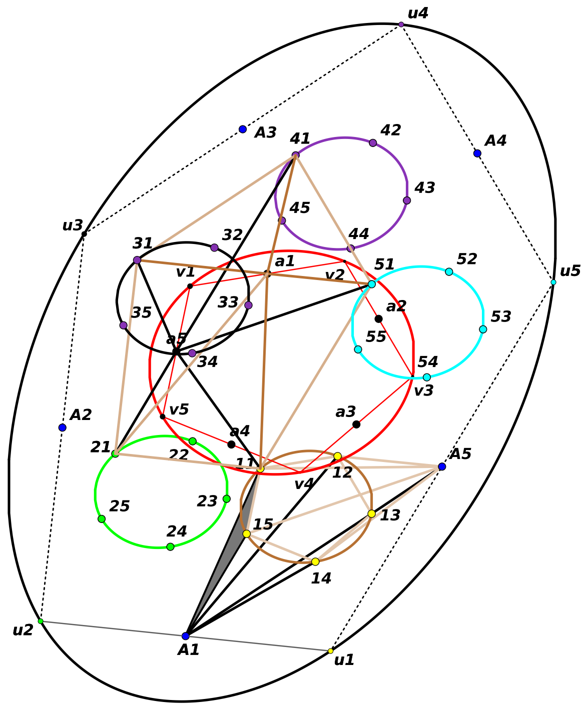
$$D_n \times D_n \cong [n] \times [n] \cong [n, 2, n] = \bullet\underset{n}{\text{———}}\bullet \qquad \bullet\underset{n}{\text{———}}\bullet . \tag{1}$$



F 2. Planar Affine Projection for $n = 5$, showing two octahedral facets

Recall the basic theorem by which the fundamental domain of a finite Coxeter group is a spherical simplex (considering its action on the unit sphere, see [14], Theorem 11.23). The fundamental tessellation belonging to the group $[n, 2, n]$ is a tessellation on $\mathbb{S}^3$ consisting of altogether $4n^2$ tetrahedra. We denote it by $\mathcal{T}$. The following properties of the fundamental tetrahedron are encoded in the Coxeter diagram of the group given in (1). It has two opposite edges of equal length, the degree of which is $2n$ in the sense that there are $2n$ tetrahedra meeting in such an edge. The other four edges are also equal to each other and are of degree 4. Hence this tetrahedron is a (spherical) *tetragonal disphenoid*, i.e. it is bounded by equal isosceles triangular facets [14]. It is symmetrical by a half-turn ϱ about the join of the midpoints of two opposite edges of degree 4. Thus ϱ induces an automorphism of the group $\bullet\underset{n}{\text{——}}\bullet \qquad \bullet\underset{n}{\text{——}}\bullet$. This automorphism interchanges the two factors in the direct product.

We obtain a tessellation $\mathcal{T}'$ on $\mathbb{S}^3$ by radially projecting P_{nn} onto this sphere. Since P_{nn} is a free sum of two regular n-gons, it has n^2 equal facets. These are tetragonal disphenoids. The (geometric) symmetry of these disphenoids is preserved through the projection, thus $\mathcal{T}'$ consists of n^2 disphenoidal tiles. Two opposite edges of such a spherical disphenoid



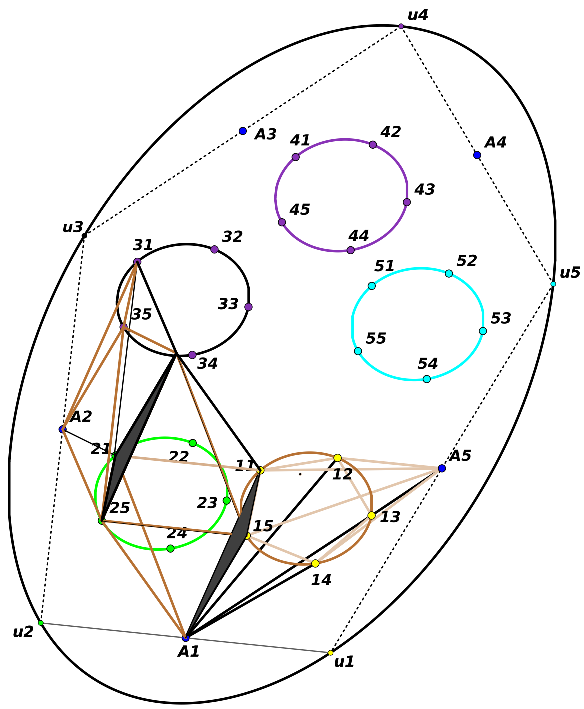
F 3. Planar Affine Projection for $n = 5$, showing two bipyramid facets

are of degree n, and the four other edges are of degree 4. Furthermore, we observe the following symmetry properties. In addition to the half-turn of the type mentioned above, a tetragonal disphenoid has mirror symmetry as well, with respect to two distinct mirror planes perpendicular to each other. Each of these planes passes through an edge while dissecting the opposite edge (these edges are those that coincide with the bases of the isosceles triangular faces). The two planes thus decompose the disphenoid into 4 equal smaller disphenoids. Thus we see that each tile of $\mathcal{T}'$ contains four of the tiles of $\mathcal{T}$. This means that $\mathcal{T}$ can be considered as a refinement of $\mathcal{T}'$.

Note, in addition, that the line of intersection of the two mirror planes serves as an axis of a half-turn to which the larger disphenoid (and hence the whole tessellation $\mathcal{T}'$) is symmetrical. The segment of this line within the disphenoid is thus a common edge of the four smaller disphenoids. We denote by γ the half-turn of this second type.

Having related to each other the tessellations $\mathcal{T}$ and $\mathcal{T}'$, it is directly seen that for a transformation that interchanges U and V the half-turn ϱ can be chosen. Thus we obtained:

$$\mathrm{Sym}(P_{nn}) \cong [n,2,n] \rtimes \langle \varrho \rangle \qquad (2)$$



F 4. Planar Affine Projection for $n = 5$, showing shared triangles between facets

for $n = 3$ and $n \geqslant 5$ (in Coxeter's notation this is the group $[[n, 2, n]]$, see [15], p. 566). The exceptional case of $n = 4$ leads to the vertex set of a regular 16-cell, whose symmetry group is larger ($= [3, 3, 4]$).

As a next step, we construct a variant of GS_n by projecting radially all the cells P^i_j, P^i and P_j onto $\mathbb{S}^3$. We regard the spherical tessellation obtained in this way as a kind of geometric realization of GS_n. We denote it by $\widehat{GS}_n$, as well as the cells by $\widehat{P}^i_j$, $\widehat{P}^i$ and $\widehat{P}_j$, respectively. In addition, we denote the spherical image of the centroid of the cell P^i_j by $\widehat{c}^i_j$. We have as well: $\widehat{u}_i \equiv u_i$ and $\widehat{v}_j \equiv v_j$. Finally, $\widehat{a}^i$, $\widehat{a}_j$ and $\widehat{b}^i_j$ denotes the spherical image of a^i, a_j and b^i_j, respectively.

Taking into account the description of GS_n given in the preceding section, the superposition of $\widehat{GS}_n$ and $\widehat{P}_{nn} = \mathcal{T}'$ shows directly that the geometric symmetry group of $\widehat{GS}_n$ remains the same as that of P_{nn}:

$$\mathrm{Sym}(\widehat{GS}_n) = \mathrm{Sym}(\widehat{P}_{nn}) \cong \mathrm{Sym}(P_{nn}). \tag{3}$$

Some properties of this group, which we shall need later as well, are as follows.

We establish that the stabilizer subgroups in $\mathrm{Sym}(\widehat{GS}_n)$ are isomorphic to

(A) $\bullet\underset{n}{\text{———}}\bullet \qquad \bullet = [n,2] \cong D_{nh}$ for $\widehat{u}_i$, $\widehat{a}^i$, $\widehat{v}_j$ and $\widehat{a}_j$;

(B) $[4,2^+] \cong D_{2d}$ for $\widehat{b}^i_j$ and $\widehat{c}^i_j$.

(Here we use the standard group notation by Coxeter and Schoenflies respectively, cf. [16], Table 2).

It follows from (A) that both $\widehat{P}^i$ and $\widehat{P}_j$ is a spherical *regular n-gonal bipyramid*, i.e. the spherical version of a 3-polytope that is composed of two equal right pyramids having a regular n-gonal basis in common. Both the geometric and combinatorial symmetry group of such a bipyramid is isomorphic to the given group. This group serves not only as the stabilizer of the points in question, but also as the stabilizer of the bipyramidal tiles containing these points in their interior.

Likewise, (B) implies that the stabilizer of $\widehat{P}^i_j$ must be the given group. However, the symmetry group of a tile of this type is larger, which is the consequence of the way as its Euclidean preimage has been constructed from a tetragonal disphenoid. Namely, this group is isomorphic to $[4,2] \cong D_{4h}$. This means that geometrically it is a (spherical) *tetragonal bipyramid*.

The case $n = 4$ is an exception again, in that both types of the cells become regular octahedra, and we obtain (the sperical image) of the regular 24-cell.

Remark 1. *Observe that in this case the construction as we obtain GS_n from P_{nn} is exactly the construction by which the regular 24-cell is obtained from the regular 16-cell through truncating its vertices* (Cèsaro's construction) [14].

Recall that here a symmetry increase occurs, namely $[3,3,4] \to [3,4,3]$.

Finally, we note that $\widehat{GS}_n$ geometrically realizes its full combinatorial symmetry, i.e. $\mathrm{Sym}(\widehat{GS}_n) \cong \mathrm{Aut}(\widehat{GS}_n)$. On the other hand, it is clear that the combinatorial symmetry is preserved through the projection procedure, i.e. $\mathrm{Aut}(\widehat{GS}_n) \cong \mathrm{Aut}(GS_n)$. Comparing this with (2) and (3), we obtain:

$$\begin{aligned} \mathrm{Aut}(GS_n) &\cong \mathrm{Sym}(\widehat{GS}_n) \cong \\ &\cong [n,2,n] \rtimes \langle \varrho \rangle \cong [[n,2,n]]. \end{aligned} \tag{4}$$

2.3. **Self-polar-duality.** The f-vector of GS_n is easily established as:

$$f(GS_n) = (n^2+2n, 6n^2, 6n^2, n^2+2n).$$

We shall see that the symmetry of the f-vector stems in fact from self-duality. Actually, we prove more, namely, that the geometric realization of GS_n on $\mathbb{S}^3$ is self-polar. We are working again in the spherical image $\widehat{GS}_n$.

We complete the notation introduced above for certain types of points as follows:

$$\widehat{c}^i = \widehat{u}_i \equiv u_i \quad \text{and} \quad \widehat{c}_j = \widehat{v}_j \equiv v_j. \tag{5}$$

We note that (A) and (B) in the preceding section justifies the following assignment of these points, and $\widehat{c}^i_j$ as well, as the (spherical) centroid of the respective bipyramidal tiles:

$$\widehat{c}^i \longleftrightarrow \widehat{P}^i, \quad \widehat{c}_j \longleftrightarrow \widehat{P}_j, \quad \widehat{c}^i_j \longleftrightarrow \widehat{P}^i_j. \tag{6}$$

We have seen above that $\mathcal{T}'$ is symmetrical to a half-turn about the axis joining the midpoints of two opposite edges of degree n of any of its tiles. From the comparison of the two tessellations above, it can be seen that the same is true for $\mathcal{T}$, concerning the edges of degree $2n$. We denote the half-turn of this latter type by β.

Consider a tile $T' \in \mathcal{T}'$ with vertex set $\{u_i, u_{i+1}, v_j, v_{j+1}\}$. Let $T \in \mathcal{T}$ be a tile contained in T' such that its vertex set is $\{u_i, \widehat{a}^i, v_j, \widehat{a}_j\} = \{\widehat{c}^i, \widehat{a}^i, \widehat{c}_j, \widehat{a}_j\}$. Then it is seen that a half-turn of type β interchanges $\widehat{a}^i$ and $\widehat{c}^i$, likewise $\widehat{a}_j$ and $\widehat{c}_j$. This amounts to saying that one apex of the n-gonal bipyramid $\widehat{P}^i$ is interchanged with its centroid, and the same happens with $\widehat{P}_j$. On the other hand, $\widehat{c}^i_j$ is interchanged with $\widehat{b}^i_j$, i.e. the centroid of $\widehat{P}^i_j$ is sent to one of its basal vertices and vice versa.

In general, we have the following correspondence:

$$\widehat{a}^{i+k} \longleftrightarrow \widehat{c}^{i-k+n}, \quad \widehat{a}_{j+l} \longleftrightarrow \widehat{c}_{j-l+n}, \quad \widehat{c}^{i+k}_{j+l} \longleftrightarrow \widehat{b}^{i-k+n}_{j-l+n}, \tag{7}$$

$k, l \in \{1, \ldots, n\}$, that is, the correspondence between the bipyramids and their vertices established locally extends to the whole structure.

Moreover, again from (A) and (B) in the preceding section follows that this correspondence induces conjugation between the respective stabilizer subgroups. This ensures that the bipyramid tiles having a vertex in common surround it according to exactly the same symmetry as the vertices surround the centroid of a corresponding bipyramid they belong to.

Thus we have proved:

Theorem 1. *$\widehat{GS}_n$ is self-polar in the sense that the transformation sending it to its dual can be realized by an isometry of order 2.*

We note that for convex 3-polytopes the analogous property has been investigated by Grünbaum et al., who call such a polyhedron *harmoniously self-dual* [1].

2.4. **A non-realizability result.** In this section we investigate whether GS_n has a polytopal realization with full symmetry. We find that the answer is negative:

Theorem 2. *For $n = 3$ and $n \geqslant 5$, GS_n cannot be realized as a boundary complex of a convex 4-polytope $\overline{GS}_n$ such that its geometric symmetry group $\mathrm{Sym}\,\overline{GS}_n$ is isomorphic to $\mathrm{Aut}\,\overline{GS}_n$, the automophism group of its face lattice.*

In proving this, we proceed indirectly. Suppose that a polytopal realization $\overline{GS}_n$ with full symmetry exists. Hence $\widehat{GS}_n$ can be considered as the spherical image of such a polytope $\overline{GS}_n$ under a radial projection. Moreover, up to isometry, $\widehat{GS}_n$ is unique in this sense:

Lemma 1. *Keeping the geometric symmetry group given in* (4) *fixed, one cannot alter the location of the vertices of the tessellation $\widehat{GS}_n$ on $\mathbb{S}^3$ without changing the action of this group on $\widehat{GS}_n$.*

Proof. Consider first the set $\left\{ \widehat{a}^{\,i}, \widehat{a}_j \mid i, j = 1, \dots, n \right\}$. As we have seen in the proof of Theorem 1, cf. the relations (5) and (7), this set is congruent to the set $U \cup V$. But is directly seen that the arrangement of the points in the latter set cannot be altered without changing its symmetry given in (2). (Equivalently, one may say as well that the convex hull of this set, being isometric with $P_{nn} = \mathrm{conv}(U \cup V)$, is a *perfect polytope* [23, 42].)

Secondly, consider the set $\left\{ \widehat{b}^{\,i}_j \mid i, j = 1, \dots, n \right\}$. A point belonging to this set is located in the midpoint of a spherical line segment $u_i v_j$. Such a line segment, being part of the intersection of two mirror planes perpendicular to each other, belongs to an axis of rotation of order four (all is meant in spherical sense). Hence these points cannot leave such axes, otherwise their number would be multiplied by 4. Neither can they be shifted within those line segments out of the midpoint positions. For, as we have seen in the preceding section, there are axes of half-turn passing through these midpoints (such half-turns are the conjugates of ϱ). So shifting to a neighbouring position would double the number of the points in question. □

This result implies that for reconstructing the polytope $\overline{GS}_n$ from this spherical image the only possibility is to locate its vertices along fixed radial straight lines.

This further implies that the shape of the bipyramid facets of $\overline{GS}_n$ is fixed as well. This is true for the facets of both type. We see it for the tetragonal bipyramid facets as follows. Fix the apices of all the facets so as to coincide with the points $\widehat{a}^i$ and $\widehat{a}_j$. Then take a tetragonal bipyramid facet $\overline{P}^i_j$, and consider its centroid $\overline{c}^i_j$. Recall that the symmetry group of a tetragonal bipyramid is isomorphic to $[4,2] \cong D_{4h}$. Then we have the following

Observation 1. *Let BP_n be an n-gonal bipyramid, i.e. a bipyramid such that its symmetry group is isomorphic to $[n,2] \cong D_{nh}$. Then its centroid can be given either as the centroid of its apices or as the centroid of its basal vertices.*

Now having fixed the apices, the only way to change the shape of this bipyramid is shifting its basal vertices along radial straight lines, all to the same extent. But such a shift would imply that the centroid in the one sense were not coinciding any more with the centroid in the other sense, which is a contradiction.

Thus we have seen that the shape of the tetragonal bipyramid facets of $\overline{GS}_n$ is uniquely determined.

Take now the other type of facets, which must be n-gonal bipyramids, with uniquely determined shape as well. Since vertices of such a bipyramid facet are completely fixed, its centroid is also fixed. Consider, say, $\widehat{c}^i$. Using Observation 1, we calculate its position in two different ways.

Let U and V be given as

$$\begin{aligned} U &= \{u_i \,|\, i = 1,2,\ldots,n\} = \left\{ \left(\cos\frac{2\pi i}{n}, \sin\frac{2\pi i}{n}, 0, 0\right) \middle| \, i = 1,2,\ldots,n \right\} \\ V &= \{v_j \,|\, j = 1,2,\ldots,n\} = \left\{ \left(0, 0, \cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}\right) \middle| \, j = 1,2,\ldots,n \right\}. \end{aligned} \tag{8}$$

For a^i and a_j we have:

$$a^i = \frac{1}{2}(u_i + u_{i+1}) \quad \text{and} \quad a_j = \frac{1}{2}(v_j + v_{j+1}),$$

for $i, j = 1,2,\ldots,n$.

We write an apex of a bipyramid facet of $\overline{GS}_n$ in the form $\overline{a}^i = \lambda_0 a^i$ and $\overline{a}_j = \lambda_0 a_j$ with some $\lambda_0 \in \mathbb{R}$. For convenience, we choose $\lambda_0 = 2$, thus we fix the apices as

$$\overline{a}^i = u_i + u_{i+1} \quad \text{and} \quad \overline{a}_j = v_j + v_{j+1} \quad \text{for} \quad i, j = 1,2,\ldots,n. \tag{9}$$

For b^i_j we have

$$b^i_j = \frac{1}{2}(u_i + v_j).$$

A basal vertex of a bipyramid facet of $\overline{GS}_n$ takes the form $\overline{b}_j^i = \lambda_1 b_j^i$ for some $\lambda_1 \in \mathbb{R}$. We determine λ_1 by applying Observation 1 for the tetragonal bipyramid facets.

Consider the facet $\overline{P}_j^i$. Its centroid $\overline{c}_j^i$ can be given on the one hand as

$$\overline{c}_j^i = \frac{1}{2}\left(\overline{a}^i + \overline{a}_j\right) = \frac{1}{2}(u_i + u_{i+1} + v_j + v_{j+1}),$$

where we applied (9). On the other hand, it can also be given as

$$\begin{aligned}\overline{c}_j^i &= \frac{1}{4}\left(\overline{b}_j^i + \overline{b}_j^{i+1} + \overline{b}_{j+1}^{i+1} + \overline{b}_{j+1}^i\right)\\ &= \frac{1}{4}\lambda_1\left(b_j^i + b_j^{i+1} + b_{j+1}^{i+1} + b_{j+1}^i\right)\\ &= \frac{1}{4}\lambda_1\left(u_i + u_{i+1} + v_j + v_{j+1}\right).\end{aligned}$$

The comparison yields $\lambda_1 = 2$. (Note that equality of λ_0 and λ_1 is consistent with the observation that P_j^i is in fact a 3-polytope even in GS_n, see Section 2.1. Thus $\widehat{P}_j^i$ is just a two times larger homothetic copy of P_j^i.) Hence we obtain for the basal vertices:

$$\overline{b}_j^i = u_i + v_j. \tag{10}$$

Now we are ready to calculate $\widehat{c}^i$ both from the apices and from the basal vertices. We denote its value obtained in the two ways by $\left(\widehat{c}^i\right)_A$ and $\left(\widehat{c}^i\right)_B$, respectively. For symmetry reasons it is sufficient to see what happens in one particular n-gonal bipyramid, thus we choose $i = 1$. From the corresponding apices we obtain:

$$\begin{aligned}\left(\widehat{c}^1\right)_A &= \frac{1}{2}\left(\widehat{a}^n + \widehat{a}^1\right) = \frac{1}{2}(u_n + 2u_1 + u_2)\\ &= \frac{1}{2}\left[(1,0,0,0) + 2\left(\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n}, 0, 0\right) + \left(\cos\frac{4\pi}{n}, \sin\frac{4\pi}{n}, 0, 0\right)\right]\\ &= \left(\frac{1}{2} + \cos\frac{2\pi}{n} + \frac{1}{2}\cos\frac{4\pi}{n}, \sin\frac{2\pi}{n} + \frac{1}{2}\sin\frac{4\pi}{n}, 0, 0\right),\end{aligned}$$

and the basal vertices yield:

$$
\begin{aligned}
\left(\widehat{c}^1\right)_B = \frac{1}{n}\sum_{j=1}^{n}\overline{b}_j^1 &= \frac{1}{n}\sum_{j=1}^{n}(u_1+v_j)\\
&= \frac{1}{n}\sum_{j=1}^{n}\left(\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n}, \cos\frac{2\pi j}{n}, \sin\frac{2\pi j}{n}\right)\\
&= \left(\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n}, 0, 0\right),
\end{aligned}
$$

where for the substitution we used (8), (9) and (10).

Take the norm of these vectors:

$$
\left\|\left(\widehat{c}^1\right)_A\right\| = 1+\cos\frac{2\pi}{n}, \qquad \left\|\left(\widehat{c}^1\right)_B\right\| = 1. \tag{11}
$$

We see that the equality holds only for $n = 4$. For $n = 3$ and $n \geqslant 5$, however, we have arrived at a contradiction, and this completes the proof of Theorem 2.

The fact that the full combinatorial symmetry group cannot be realized by affine symmetries of a combinatorially prescribed polytope does not occur in 3 dimensions [33]. The first observation of this phenomenon in 4 dimensions is due to Bokowski, Ewald, and Kleinschmidt [7]. The open problem of McMullen [6] is of the same kind. Smith Theory implies that in cases where a complex is not realizable with full symmetry, the realization space is not contractible.

2.5. **Polytopality of GS_n.** The self-duality of GS_n provides an alternative way of obtaining it, by starting from a dual polytope and applying a construction that is a dual of ours described above. We exemplify this through a historically interesting instance. Recall that the regular 24-cell can be constructed not only by Cèsaro's method but by *Gosset's construction* as well: in contrast to Cèsaro, who cuts pyramids from the corners of the regular 16-cell, Gosset, dually, erects pyramids on the facets of the 4-cube (cf. Remark 1 above and Coxeter [14], p. 150). (A closely related but a much simpler example in dimension 3 is the way as the *rhombic dodecahedron* is constructed from the cube [14], p. 26).)

More generally, instead of the *free sum* of two regular n-gons of *equal* size, one can start from the *product* $C_m \times C_n$ of two regular polygons such that neither their size nor the number of their sides is not necessarily equal. We denote this 4-polytope, which is the dual of our P_{nn} above for $m = n$, by Q_{mn}. Now apply a method, which is called an *E-construction*, performed in two steps as follows:

(1) stellarly subdivide all facets of Q_{mn},
(2) merge facets of the subdivision sharing a 2-face of Q_{mn}.

The E-construction was introduced by Eppstein, Kuperberg and Ziegler in 2003 in order to obtain 2-simple, 2-simplicial 4-polytopes [18]. It was soon extended to arbitrary dimensions and to spheres and lattices by Paffenholz and Ziegler [38]. In this line of research, which is independent of ours, it turned out that the E-construction yields GS_n as a special case, as we just outlined [46].

For arbitrary $m, n \geqslant 3$, denote the corresponding CW-spheres that the E-construction yields by $E(Q_{mn})$. In [46] Ziegler has given a proof that they are combinatorially self-dual (his proof and our proof for Theorem 1 above closely resemble, necessarily, each other; cf. Theorem 4.1 in [36]).

In this more general setting, one can prove the following interesting result.

Theorem 3. (Paffenholz [36]) *The CW-spheres $E(Q_{mn})$ are polytopal for all $m, n \geqslant 3$.*

In the special case

$$\frac{1}{m} + \frac{1}{n} \geqslant \frac{1}{2} \tag{12}$$

the first proof was given by Francisco Santos. It is known from a personal communication 2003 by him, cited in [36] and [46]. This special case has been treated also by Ziegler in [46], where he applied a certain generalization of the construction given by Santos for $E(Q_{33})$. Ziegler has given coordinates here as well.

Andreas Paffenholz in [36] has investigated in detail the polytopal realizations of $E(Q_{mn})$. In particular, he proved as well that the projective realization space of $E(Q_{33})$ is at least nine-dimensional and that of $E(Q_{44})$ at least four-dimensional (the latter result implies that the 24-cell is not projectively unique). He established as well that for all polytopes P realizing $E(Q_{mn})$ with relatively prime $m, n \geqslant 5$ the combinatorial symmetry group $\mathrm{Aut}(P)$ is greater than the geometric symmetry group $\mathrm{Sym}(P)$. Moreover, even the group $\mathbb{Z}_m \times \mathbb{Z}_n$, which is always contained as a subgroup in $\mathrm{Aut}(P)$, can be geometrically realized only in the five cases $(m, n) = (3,3), (3,4), (3,5), (4,4), (3,6)$ allowed by condition (12) (up to interchanging m and n). For further details see the Ph.D. thesis of Paffenholz [37].

3. C -

The task of finding a matroid polytope consistent with a certain boundary complex can be viewed as a constraint satisfaction problem where the orientations of bases are the variables, and the constraints are the chirotope axioms of oriented matroids (see e.g. [2]), along with certain equations induced by the boundary structure. In the uniform case, this results in a variation of the well known Boolean Satisfiability (or SAT) problem [21], where the constraints are the 3-term Grassmann-Plücker relations (which can be encoded in SAT as ternary exclusive-ors of binary exclusive-ors), along with the aforementioned equalities (which can be viewed as a certain projection of the Grassmann-Plücker relations). Although SAT is the canonical NP-Complete problem, variations on the heuristic backtracking procedure of Davis-Putnam-Logemann-Loveland (DPLL) [17] have achieved reasonable success for certain classes of problems. In this subsection we describe a DPLL-like algorithm (and implementation) for the generation of matroid polytopes in the non-uniform case. This requires the use of a three valued "logic" of signs, and some additional constraints as compared to the uniform case.

The key idea of our constraint satisfaction algorithm is that of *variable forcing*. This is based on the observation that if all but one of the variables in a disjunction are fixed, and the disjunction is not yet satisfied, then the value of the last variable is forced. This forcing is known as *unit propagation* in the DPLL context. In the constraint satisfaction algorithm implemented in `nuoms` [12] a slightly more general inference system is used, since for each variable we maintain what subset of $\{-1,0,+1\}$ is still possible; furthermore the clauses are slightly more complex than disjunctions. Nonetheless it is possible after setting the value of a variable in a clause to deduce constraints on the values of the remaining variables (in the best case forcing them to a particular value). We discuss certain optimizations of the variable forcing process after introducing the various types of clauses.

There are four types of constraints used in `nuoms`: the *boundary constraints* induced by the boundary complex, the *convexity constraints* to insure no (relative) interior points are present in final result, the *matroid constraints* that insure that the basis exchange axiom is not violated, and the 3-term *Grassmann-Plücker* constraints. They are checked in this order, roughly in order of increasing effort. In the uniform case, neither the convexity constraints (implied by the boundary constraints) nor the matroid constraints (implied by the uniformity condition) are necessary.

The boundary constraints are a set of equalities derived from the input boundary complex. Each top dimensional set F of the input boundary complex defines a *facet* of the resulting matroid polytope in the following sense. In order to avoid interior points, it must be the case that for all boundary simplices, i.e. $(r-1)$-sets $G \subseteq F$, for all $i, j \notin G$, $[Gi] = [Gj]$. Thus for each input facet, we derive at least $n-r-1$ equalities where r is the rank and n is the total number of elements (in general of course many more equalities are implied by transitivity). It may be desirable to apply a preprocessing step to eliminate sign variables using these equalities. In order to support working with partial and dynamically discovered boundary information (more interesting in the context [13] where the uniform code was developed), we consider a constraint of the form that insists that the signs of bases derived as single element extensions of boundary simplices are *monotone*, i.e. do not contain both positive and negative signs. For problems currently approachable by constraint satisfaction, this constraint, and any forcing of variables, can be checked very efficiently via the use of bitmasks.

The convexity constraints check a combinatorial analogue of Caratheodory's theorem for each element. Although in the uniform case it suffices to establish that each element is contained in some facet, in the non-uniform case, the lower dimensional structure of the facets can be more complicated, so some kind of further constraint is necessary. These are again reduced to checking for each element that a certain set of signs is monotone, and the implementation is similar to checking the boundary constraints.

In order to simplify checking that the basis exchange axiom is satisfied, we employ an observation of Guedes de Oliveira (see [2], Ex. 3.21) that instead of checking every pair of non-zero bases, it suffices to know that there exists some non-zero *root basis* B_0 such that for every other basis B', there exists a pivot that moves closer to B_0. This amounts to computing a spanning tree of the graph of all pivots and is much faster, although it requires a non-zero basis be known. The existence of a non-zero neighbouring basis to each basis can also be checked in a constant number of operations using bitmasks.

In general the largest set of constraints is the 3-term Grassmann-Plücker relations. Since these constraints are so numerous, but on the other hand involve only six basis signs, we take some care to process them efficiently. The permissable values are for each variable are stored as a three bit mask, and the state of a given Grassmann-Plücker relation is then encoded into a single 18 bit number. A simple (although a bit large) state machine is then used to decide given the current state, and a new assignment, what is the new state, and what if anything is forced.

The inductive structure of the algorithm is as follows. Given a partial chirotope, choose some unoriented basis, and orient it. Find all variables forced (constrained) by the previously described constraints. If this does not yield a contradiction, repeat. In order to systematically try all possible orientations, a stack is used in the manner of the standard depth first search of a graph. When an unoriented basis is examined, all possible orientations are pushed onto the stack, and after the program has explored one setting in a depth first manner (i.e. tried to either complete a chirotope or derive a contradiction), it returns to the stack to retrieve the other possibilities.

Using the software [12] described in this section, we were able to find several million matroid polytopes consistent with the spheres described in Section 2 after a computation of only a few seconds on a current desktop PC.

4. I C H

Hyperline arrangements have previously been used by Bokowski and Guedes de Oliveira [8] to generate uniform oriented matroids (for complete discussion of hyperline arrangements see [5]). In this section we describe, via some Haskell [25] code, the generalization to the non-uniform case. In general a rank r hyperline arrangement on $n-1$ elements consists of all rank 2 contractions of some oriented matroid. Each rank 2 contraction (i.e. a row in Figure 5) is represented by the contracted elements ($\triangle$), along with a *hyperline sequence*, i.e. an oriented matroid of rank 2 (ω), represented as a signed permutation. To extend this to an n element oriented matroid, we need to insert the element n either into the signed permutation, or, in the non-uniform case, possibly into the set of contracted elements. In order to simplify the presentation, we restrict our attention to the rank 5 case.

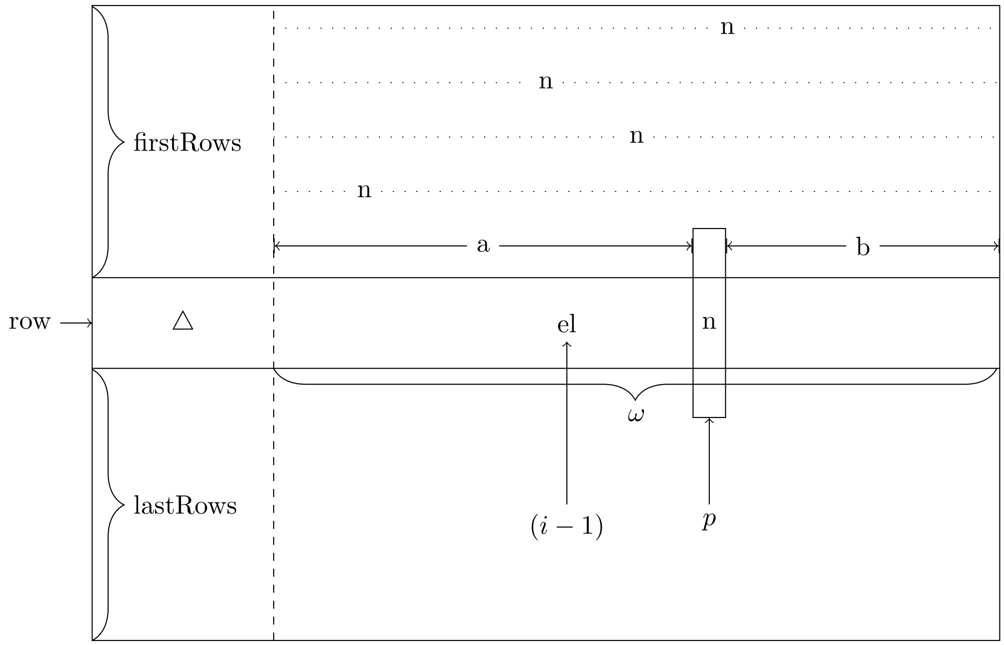
For a hyperline sequence representation of a rank 5 matroid polytope we can use a special data structure. Our convexity requirement implies that we have never three elements within one line. Moreover, each 2-dimensional affine hull of vertices of a convex polytope is convex again. This implies that we can assume that each hyperline is that of a planar n-gon. The corresponding rank 2 oriented matroid can be descibed via the cyclic order of these elements. We can store the rank 5 oriented matroid as a list of pairs of k-gons, with $k \geqslant 3$ depending on the hyperline, together with rank 2 oriented matroids, the hyperline sequences, i.e., rank 2 contractions at these hyperlines. The latter has a circular structure and we can assume to have the smallest element with positive sign within their first set. For the n-gons we can also assume that their lists begin with the smallest element of each n-gon. For one row in the hyperline representation we obtain a 5-tuple with a zero sign when either two elements belong to the same list within the rank 2 oriented matroid or when we can choose four elements within the first component of that row.

4.1. **The uniform case.**

Before delving into the non-uniform case, we first recall extending a hyperline arrangement by one element in the uniform case.

4.1.1. *The function `inRow` in the uniform case.*

The top level function in the extension algorithm is *inRow*. The variable *hyp* represents the list of all



F 5. Inserting element n into the the hyperline structure (uniform case)

hyperline sequences that we extend row by row. The variable χ represents the signs of all abstract simplices that we know so far.

$$
\begin{array}{l}
\mathit{inRow} :: \mathit{Int} \to \mathit{Int} \to ([(\mathit{Ngon}, \mathit{OM2})], [\mathit{Or}]) \to [([(\mathit{Ngon}, \mathit{OM2})], [\mathit{Or}])] \\
\mathit{inRow}\ n\ \mathit{row}\ (\mathit{hyp}, \chi) = \\
\quad [((\mathit{firstRows} +\!\!+ [(\triangle, \mathit{ext}\ s\ n\ p\ \omega)] +\!\!+ \mathit{lastRows}), \mathit{newsigns}) \\
\qquad \mid s \leftarrow [-1, 1], p \leftarrow [1 \mathinner{..} |\omega|], \\
\qquad \mathit{newsigns} \leftarrow \mathbf{let}\ \mathit{st} = [\mathit{norm}\ (\triangle +\!\!+ [\omega\, !!\, (i-1), s * n], 1) \mid i \leftarrow [1 \mathinner{..} p]] \\
\qquad\quad +\!\!+ [\mathit{norm}\ (\triangle +\!\!+ [s * n, \omega\, !!\, (i-1)], 1) \mid i \leftarrow [(p+1) \mathinner{..} |\omega|]] \\
\qquad\quad \mathbf{in}\ \mathit{newOrEmpty}\ n\ \chi\ \mathit{st}] \\
\quad \mathbf{where}\ (\mathit{firstRows}, ((\triangle, \omega) : \mathit{lastRows})) = \mathit{splitAt}\ (\mathit{row} - 1)\ \mathit{hyp} \\
\qquad |\omega| = \mathit{length}\ \omega
\end{array}
$$

For the most part, the notation used by Haskell is standard. We mention only a few notational aspects here (for further details, see e.g. [25]).

(1) Lists are denoted by [ ], tuples by ()
(2) $a +\!\!+ b$ denotes the concatenation of lists a and b
(3) $l\, !!\, n$ denotes item n in list l
(4) $h : t = l$ denotes the decomposition of list l into first element h and remainder t.

We interpret the pair of a list of integers and a sign (where 2 indicates unknown) as an oriented (abstract) simplex. The function *norm* returns such an oriented simplex with positive and sorted elements whereby the sign has changed accordingly.

$$\begin{aligned}
&\mathit{norm} :: \mathit{OB} \to \mathit{OB}\\
&\mathit{norm}\ (\mathit{tu}@(h : \mathit{rest}), s) = \mathit{normPos}\ (\mathit{list}, s * \mathit{signum}\ \mathit{prod})\\
&\textbf{where}\ \mathit{prod} = \mathit{product}\ \mathit{tu}; \mathit{list} = \mathit{map}\ \mathit{abs}\ \mathit{tu}\\
&\quad \mathit{normPos} :: \mathit{OB} \to \mathit{OB}\\
&\quad \mathit{normPos}\quad (\mathit{tuple}@(h : \mathit{rest}), \mathit{sign})\\
&\qquad \mid \mathit{rest} \equiv [\,] = ([h], \mathit{sign})\\
&\qquad \mid h \equiv \mathit{minimum}\ \mathit{tuple} = ([h] +\!\!+ \mathit{fst}\ \mathit{next}, \mathit{snd}\ \mathit{next})\\
&\qquad \mid \mathit{odd}\ (\mathit{length}\ \mathit{rest}) = \mathit{normPos}\ (\mathit{rest} +\!\!+ [h], -\mathit{sign})\\
&\qquad \mid \mathit{otherwise} = \mathit{normPos}\ (\mathit{rest} +\!\!+ [h], \mathit{sign})\\
&\qquad\quad \textbf{where}\ \mathit{next} = \mathit{normPos}\ (\mathit{rest}, \mathit{sign})
\end{aligned}$$

Splitting *hyp* at the position *row* leads to the current row data structure $(\triangle, \omega)$ as the head of the second component of list returned by the function *splitAt*. We insert the new signed element $s \times n$ in ω by using the function *ext* (described below) in all possible ways. The variable *st* stores the list of new signs that we know after the insertion has been completed in this row. We compare the new signed element $s \times n$ with all other elements in this row to obtain new signs of abstract simplices. When we cannot pick *newsigns*, we do not get an extension. This occurs when we have a sign contradiction that will be detected in the function *newOrEmpty*.

$$\begin{aligned}
&\mathit{newOrEmpty} :: \mathit{Int} \to [\mathit{Or}] \to [(\mathit{Tu}, \mathit{Or})] \to [[\mathit{Or}]]\\
&\mathit{newOrEmpty}\ n\ \chi\ [\,] = [\chi]\\
&\mathit{newOrEmpty}\ n\ \chi\ ((\mathit{tu}, s) : \mathit{rest})\\
&\quad \mid e \notin [s, 2] = [\,]\\
&\quad \mid \mathit{otherwise} = \mathit{newOrEmpty}\ n\ \mathit{newChi}\ \mathit{rest}\\
&\quad \textbf{where}\ i = \mathit{head}\ (\mathit{elemIndices}\ \mathit{tu}\ (\mathit{tailTup}\ n))\\
&\qquad (a, (e : b)) = \mathit{splitAt}\ i\ \chi; \mathit{newChi} = a +\!\!+ [s] +\!\!+ b
\end{aligned}$$

This function compares the preliminary sign list χ with *st*. Within the list *tupels* 5 *finalN* we determine the position *i* of the actual tuple *tu* and we find the corresponding sign *e*. When this sign *e* is different from 2, i.e., it has been determined, and when it is not equal to *s*, we obtain a contradiction, i.e., the result is the empty set. Thus the function *inRow* leads to a list of all extensions within the row under consideration together with new signs that are compatible with the given sign vector.

The function *tailTup* returns all new 5-tuples that occur the first time when we have *n* as the new element, i.e., all 5-tuples with *n* at the end. At first the signs of all $\binom{n}{5}$ signed bases are considered to be unknown, i.e., *signs* provided such a list with entries 2.

The details of inserting a signed element into a hyperline sequence are taken care of by the function *ext*. It determines for a sign s of the new element n, its position p and for the uniform rank 2 contraction along the hyperline its one element extension.

$$
\begin{aligned}
&\mathit{ext} :: \mathit{Int} \to \mathit{Int} \to \mathit{Int} \to \mathit{OM2} \to \mathit{OM2}\\
&\mathit{ext}\ s\ n\ p\ \omega = a +\!\!+ [[s * n]] +\!\!+ b\\
&\quad \textbf{where}\ (a,b) = \mathit{splitAt}\ p\ \omega
\end{aligned}
$$

We do not discuss the frame that is still missing to apply this kernel structure *inRow* repeatedly and that does the next extension when we do not extend the matroid polytope by just one element.

4.2. **The non-uniform case.** We now discuss the changes needed to extend a hyperline configuration in the non-uniform case.

The function *inRow* now has two cases, depending on whether we insert the new element into the hyperline *gon*, or into the corresponding rank 2 oriented matroid ω.

$$
\begin{aligned}
&\mathit{inRow} :: \mathit{Int} \to \mathit{Int} \to ([\mathit{OM5}],[\mathit{Or}]) \to [([\mathit{OM5}],[\mathit{Or}])]\\
&\mathit{inRow}\ n\ \mathit{row}\ \mathit{pair} = (\mathit{inHl}\ n\ \mathit{row}\ \mathit{pair}) +\!\!+ (\mathit{inOM2}\ n\ \mathit{row}\ \mathit{pair})
\end{aligned}
$$

The case of inserting into the rank 2 oriented matroid is analogous to the uniform case of *inRow*, with the distinction that each position may have a set of elements.

$$
\begin{aligned}
&\mathit{inOM2} :: \mathit{Int} \to \mathit{Int} \to ([\mathit{OM5}],[\mathit{Or}]) \to [([\mathit{OM5}],[\mathit{Or}])]\\
&\mathit{inOM2}\ n\ \mathit{row}\ (\mathit{rows},\chi)\\
&\quad = [((\mathit{firstRows} +\!\!+ [(\mathit{gon}, \mathit{ext}\ s\ n\ p\ q\ \omega)] +\!\!+ \mathit{lastRows}),\\
&\qquad \mathit{newOrEmpty}\ n\ \chi\ (\mathit{newSigns}\ q\ \mathit{gon}\ \omega))\\
&\qquad \mid s \leftarrow [-1,1], p \leftarrow [1\,..\,|\omega|], q \leftarrow [0,1],\\
&\quad \textbf{where}\ (\mathit{firstRows},((\mathit{gon},\omega):\mathit{lastRows})) = \mathit{splitAt}\ (\mathit{row}-1)\ \mathit{rows}\\
&\qquad |\omega| = \mathit{length}\ \omega;
\end{aligned}
$$

We omit here basic functions *tuples*, *tuplesL*. The function *tuples* returns all r-tuples of the list of the first n natural numbers and the function *tuplesL* returns all r-tuples of any given list of integers.

$$
\begin{array}{l}
\mathit{newSigns} :: \mathit{Int} \to \mathit{Ngon} \to \mathit{OM2} \to [(\mathit{Tu},\mathit{Or})] \\
\mathit{newSigns}\ \mathit{between}\ \mathit{gon}\ \omega \\
\quad \mid \mathit{between} \equiv 0 = \\
\qquad [\mathit{norm}\ (\triangle +\!\!+ [\omega\,!!\,(i-1), s*n], 1) \mid i \leftarrow [1\,..\,p-1], \triangle \leftarrow \mathit{trs}] \\
\qquad +\!\!+ [\mathit{norm}\ (\omega\,!!\,(p) +\!\!+ [\omega\,!!\,(p), s*n], 0)] \\
\qquad\quad +\!\!+ [\mathit{norm}\ (\triangle +\!\!+ [s*n, \omega\,!!\,(i-1)], 1) \mid i \leftarrow [p+1\,..\,|\omega|], \triangle \leftarrow \mathit{trs}] \\
\qquad\qquad +\!\!+ [\mathit{norm}\ (\square +\!\!+ [n], 0) \mid \square \leftarrow \mathit{tuplesL}\ 4\ \mathit{gon}] \\
\quad \mid \mathit{between} \equiv 1 = \\
\qquad [\mathit{norm}\ (\triangle +\!\!+ [\omega\,!!\,(i-1), s*n], 1) \mid i \leftarrow [1\,..\,p], \triangle \leftarrow \mathit{trs}] \\
\qquad +\!\!+ [\mathit{norm}\ (\triangle +\!\!+ [s*n, \omega\,!!\,(i-1)], 1) \mid i \leftarrow [p+1\,..\,|\omega|], \triangle \leftarrow \mathit{trs}] \\
\qquad\quad +\!\!+ [\mathit{norm}\ (\square +\!\!+ [n], 0) \mid \square \leftarrow \mathit{tuplesL}\ 4\ \mathit{gon}] \\
\quad \textbf{where} \\
\qquad \mathit{trs} = \mathit{tuplesL}\ 3\ \mathit{gon}
\end{array}
$$

The function *inHl* considers the various ways to insert into the hyperline (convex polygon).

$$
\begin{array}{l}
\mathit{inHl} :: \mathit{Int} \to \mathit{Int} \to ([\mathit{OM5}],[\mathit{Or}]) \to [([\mathit{OM5}],[\mathit{Or}])] \\
\mathit{inHl}\ n\ \mathit{row}\ (\mathit{rows},\chi) \\
\quad = [((\mathit{firstRows} +\!\!+ [(\mathit{take}\ g\ \mathit{gon} +\!\!+ [n] +\!\!+ \mathit{drop}\ g\ \mathit{gon}, \omega)] \\
\qquad +\!\!+ \mathit{lastRows}), \mathit{signs}) \\
\qquad \mid g \leftarrow [1\,..\,\mathit{length}\ \mathit{gon}], \\
\qquad \mathit{signs} \leftarrow \textbf{let}\ \mathit{si} = [\mathit{norm}\ (\triangle +\!\!+ [p_1,p_2], 1) \\
\qquad\quad \mid \triangle \leftarrow \mathit{tuplesL}\ 3\ (\mathit{take}\ g\ \mathit{gon} +\!\!+ [n] +\!\!+ \mathit{drop}\ g\ \mathit{gon}), \\
\qquad\quad n \in \triangle, [p_1,p_2] \leftarrow \mathit{pairs}\ \omega] \\
\qquad +\!\!+ [\mathit{norm}\ (\triangle +\!\!+ [p_1,p_2], 0) \\
\qquad\quad \mid \triangle \leftarrow \mathit{tuplesL}\ 3\ (\mathit{take}\ g\ \mathit{gon} +\!\!+ [n] +\!\!+ \mathit{drop}\ g\ \mathit{gon}), \\
\qquad\quad n \in \triangle, u \leftarrow [1\,..\,\mathit{length}\ \omega], \\
\qquad\quad [p_1,p_2] \leftarrow \mathit{tuplesL}\ 2\ (\omega\,!!\,(u-1))] \\
\qquad +\!\!+ [\mathit{norm}\ (\triangle +\!\!+ [n,x], 0) \mid \triangle \leftarrow \mathit{tuplesL}\ 3\ \mathit{gon}, \\
\qquad\quad n \notin \triangle, u \leftarrow [1\,..\,\mathit{length}\ \omega], \\
\qquad\quad x \leftarrow (\omega\,!!\,(u-1)) +\!\!+ (\mathit{gon} \setminus\!\setminus \triangle)] \\
\qquad \textbf{in}\ \mathit{newOrEmpty}\ n\ \chi\ \mathit{si}] \\
\quad \textbf{where}\ (\mathit{firstRows}, ((\mathit{gon},\omega) : \mathit{lastRows})) = \mathit{splitAt}\ (\mathit{row}-1)\ \mathit{rows} \\
\qquad \mathit{pairs} :: \mathit{OM2} \to \mathit{OM2} \\
\qquad\quad \mathit{pairs}\ \omega = [[x,y] \mid [u,v] \leftarrow \mathit{tuples}\ 2\ (\mathit{length}\ \omega), \\
\qquad\qquad x \leftarrow \omega\,!!\,(u-1), y \leftarrow \omega\,!!\,(v-1)]
\end{array}
$$

The former function *ext* has now two cases (specified by the flag q) depending of whether we insert the new element at position p within an already existing list or as a new single element list between two lists.

$$
\mathit{ext} :: \mathit{Int} \to \mathit{Int} \to \mathit{Int} \to \mathit{Int} \to \mathit{OM2} \to \mathit{OM2}
$$

$$
\begin{aligned}
&\mathit{ext}\ s\ n\ p\ q\ \omega \\
&\quad |\ q \equiv 0 = \mathit{take}\ (p-1)\ a +\!\!+ [(\mathit{last}\ a) +\!\!+ [s * n]] +\!\!+ b \\
&\quad |\ q \equiv 1 = a +\!\!+ [[s * n]] +\!\!+ b \\
&\quad \textbf{where}\ (a,b) = \mathit{splitAt}\ p\ \omega
\end{aligned}
$$

The method of the second author of using his specific SAT solver for finding matroid polytopes was much faster than the Haskell based algorithm of the first author. However, different methods cast new light on each other and facilitate the checking of results.

In this context we mention that Schewe [44] has used successfully existing SAT solvers different from that of the second author to decide the embeddability of certain 2-manifolds and the realizability of certain point-line configurations.

A

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R

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