# INNER DIAGONALS OF CONVEX POLYTOPES

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*To Branko Grünbaum in honor of his seventieth birthday*

ABSTRACT.

An *inner diagonal* of a polytope $P$ is a segment that joins two vertices of $P$ and that lies, except for its ends, in $P$'s relative interior. The paper's main results are as follows: (a) Among all $d$-polytopes $P$ having a given number $v$ of vertices, the maximum number of inner diagonals is $\binom{v}{2} - dv + \binom{d+1}{2}$; when $d \geq 4$ it is attained if and only if $P$ is a stacked polytope. (b) Among all $d$-polytopes having a given number $f$ of facets, the maximum number of inner diagonals is attained by (and, at least when $d = 3$ and $f \geq 6$, only by) certain simple polytopes. (c) When $d = 3$, the maximum in (b) is determined for all $f$; when $f \geq 14$ it is $2f^2 - 21f + 64$ and the unique associated $p$-vector is $5^{12}6^{f-12}$. (d) Among all simple 3-polytopes with $f$ facets, the minimum number of inner diagonals is $f^2 - 9f + 20$; when $f \geq 9$ the unique associated $p$-vector is $3^2 4^{f-4}(f-1)^2$ and the unique associated combinatorial type is that of the wedge over an $(f-1)$-gon.

## INTRODUCTION

For $1 \leq i \leq d$, an *$i$-diagonal* of a $d$-polytope $P$ is a segment $[x, y]$ whose ends are vertices of $P$ and whose *carrier* in $P$ (i.e., the smallest face of $P$ that contains $[x, y]$) is of dimension $i$. The number of $P$'s $i$-diagonals is denoted by $\delta_i(P)$. Thus $\delta_1(P)$ is the number of $P$'s edges and $\delta_d(P)$ is the number of $P$'s *inner diagonals*. The focus here is on $\delta_d$.

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It seems strange that, except for $d = 2$, so little attention has been paid to inner diagonals of $d$-polytopes. There is an interesting paper of Mani [Ma] and an attractive open problem of von Stengel [vS] (see Section 3), but we know of little else. Following some preliminaries in Section 1, our main results appear in Section 2. They consist of the determination, for each $f \geq 4$, of both the minimum and the maximum of $\delta_3(P)$ as $P$ ranges over all simple 3-polytopes that have precisely $f$ facets. Section 3 contains some results on inner diagonals of higher-dimensional polytopes, including the determination for all $d$ of the maximum number of inner diagonals of $d$-polytopes having a given number of vertices.

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## 1. Preliminaries

We use $v(P)$ and $f(P)$ to denote respectively the number of vertices and the number of facets ($(d-1)$-faces) of a $d$-polytope $P$. A $d$-polytope is *simplicial* if each of its facets is a simplex, and it is *simple* if each of its vertices is incident to precisely $d$ edges (equivalently, to precisely $d$ facets). It seems appropriate, in each dimension $d$, to focus on general $d$-polytopes having a given number of vertices or of facets, and to focus further on simplicial $d$-polytopes with a given number of vertices and on simple $d$-polytopes with a given number of facets. The latter foci are motivated by the fact that each $d$-polytope $P$ can be approximated arbitrarily closely by simplicial $d$-polytopes with $v(P)$ vertices and by simple $d$-polytopes with $f(P)$ facets.

We say that two vertices of a polytope are *estranged* if they do not share (i.e., do not lie together on) any facet, and that two facets are *estranged* if they do not share any vertex (i.e., they are disjoint). Note that two vertices are joined by an inner diagonal if and only if they are estranged. Note also that under the standard polarity operation for polytopes, simple polytopes correspond to simplicial polytopes, the vertices of a polytope correspond to the facets of its polar, and estranged pairs of vertices in a polytope correspond to estranged pairs of facets in its polar.

The following two remarks are immediate consequences of the basic definitions.

**1.1. Remark.** *If a $d$-polytope $P$ has $v(P)$ vertices, then*

$$\delta_1(P) + \delta_2(P) + \cdots + \delta_d(P) = \binom{v(P)}{2},$$

*with* $\delta_1(P) + \delta_d(P) = \binom{v(P)}{2}$ *when* $P$ *is simplicial.*

**1.2. Remark.** *For a* $d$*-polytope to have at least one inner diagonal, its number of vertices must be at least* $d + 2$ *and its number of facets must be at least* $2d$*.*

*Proof.* If a $d$-polytope has only $d + 1$ vertices, it is a simplex and has no inner diagonal. However, for each $n \geq d + 2$ there is a simplicial $n$-vertex $d$-polytope that has an estranged pair of vertices and hence has an inner diagonal. (Start, for $n = d + 2$, with a bipyramid over a $(d - 1)$-simplex.)

Each vertex of a $d$-polytope is incident to at least $d$ facets. Hence each vertex of an estranged vertex-pair is incident to at least $d$ facets that are not incident to the other vertex of the pair. □

A $d$-polytope $P$ is called a *pyramid* if there is a facet $F$ of $P$ that contains all but one vertex of $P$. Each facet that has this property is called a *base* of the pyramid and the remaining vertex is called an *apex*. Note that for each choice of $d \geq 3$, $v > d$, and $f > d$, there is a $d$-pyramid that has $v$ vertices and there is a $d$-pyramid that has $f$ facets. Note also that when $d \geq 2$ a $d$-pyramid has no estranged pair of facets and also no estranged pair of vertices.

When $Q$ is a $(d - 1)$-polytope in $\mathbb{R}^{d-1}$ and $F$ is a facet of $Q$, a *wedge over* $Q$ *with foot* $F$ is a $d$-polytope $W \subset \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ that is obtained by intersecting the cylinder $Q \times [0, \infty[$ with a halfspace $J$ in $\mathbb{R}^d$ such that $J$ contains $Q \times \{0\}$, $J$'s bounding hyperplane $H$ contains $F \times \{0\}$, and $H$ intersects the open ray $\{x\} \times ]0, \infty[$ for each $x \in Q \setminus F$. Combinatorially, $W$ may be obtained from the product $Q \times [0, 1]$ by (in effect) collapsing $F \times [0, 1]$ to $F \times \{0\}$.

We use the following easily verified property of wedges.

**1.3. Remark.** *Suppose that* $Q$ *is a simple* $(d - 1)$*-polytope with* $f - 1$ *facets, that* $F$ *is a facet of* $Q$*, and that* $W$ *is a wedge over* $Q$ *with foot* $F$*. Then* $W$ *is a simple* $d$*-polytope with* $f$ *facets, and the number of inner diagonals of* $W$ *is twice the number of inner diagonals of* $Q$ *that miss* $F$*.*

For other basic properties of polytopes, the reader is referred to the books of Grünbaum [Gr1] and Ziegler [Zi].

## 2. Inner diagonals of 3-polytopes

This section is concerned with the combinatorial structure of 3-polytopes, or, in view of Steinitz's theorem [SR], [Gr1], with 3-connected planar graphs. The facets of a 3-polytope correspond to the nonseparating simple cycles in its graph, and two vertices are joined by an inner diagonal if and only if they do not lie together on any such cycle. We shall continue to use the geometric language since it is from a geometric viewpoint that the inner diagonals are of special interest.

**2.1. Lemma.** *With $i \geq 4$, suppose that $F$ is an $i$-gonal facet of a 3-polytope $P$ and $G'$ is a graph obtained from that of $P$ by adding an edge joining two nonadjacent vertices of $F$. Then there is a 3-polytope $P'$ whose graph is $G'$, and in each such $P'$ the number of estranged pairs of vertices exceeds that of $P$.*

*Proof.* Adding an edge as described preserves both planarity and 3-connectedness. Vertices of $F$ that are on opposite sides of the added edge are estranged in $P'$ but not in $P$. □

**2.2. Theorem.** *For each 3-polytope $P$ with $v$ vertices,*

$$\delta_3 \leq \frac{1}{2}(v^2 - 7v + 12),$$

*with equality if and only if $P$ is simplicial.*

*Proof.* It follows from Lemma 2.1 that as $P$ ranges over all 3-polytopes with $v$ vertices, the maximum number of inner diagonals is attained only by certain simplicial polytopes. For each simplicial 3-polytope with $v$ vertices, it follows from Euler's theorem and a double counting of incidences that $2v + 2f - 4 = 2\delta_1 = 3f$, whence $f = 2v - 4$, $\delta_1 = 3v - 6$, and

$$\delta_3 = \binom{v}{2} - (3v - 6) = \frac{1}{2}(v^2 - 7v + 12).$$

□

**2.3. Lemma.** *With $i \geq 4$, suppose that $F$ is an $i$-gonal facet of a 3-polytope $P$, $G'$ is a graph obtained from that of $P$ by adding an edge joining two nonadjacent vertices $p$ and $q$ of $F$, and $P'$ is a polytope whose graph is $G'$. Suppose also that $i \geq 5$ or that $i = 4$ and there is a facet $Y$ of $P$ that misses both $p$ and $q$. Then in $P'$ the number of estranged pairs of facets exceeds that of $P$.*

*Proof.* When $i \geq 5$, let $E$ be an edge of $F$ that misses both $p$ and $q$ and let $Y$ be the unique facet of $P$ such that $F \cap Y = E$. In both cases, $Y$ is disjoint from one of the facets (call it $Z$) into which $F$ is split, and hence the pair $\{Y, Z\}$ contributes to the number of estranged pairs of facets of $P'$ while $\{Y, F\}$ did not contribute to that number for $P$. □

**2.4. Theorem.** *Suppose that $v \geq 4$ and that $P$, among all 3-polytopes with $v$ vertices, is one that has the maximum number of estranged pairs of facets. Then $P$ is simplicial or is a pyramid over a 4-gon.*

*Proof.* In view of Lemma 2.3, it suffices to consider the case in which the following three conditions are satisfied: (a) $P$ has a 4-gonal facet $F$; (b) each facet of $P$ has at most 4 vertices; (c) for each facet $Y \neq F$, $F \cap Y$ consists of an edge of $F$.

If some such $Y$ is a 4-gon, then by appropriate addition of a diagonal to each of $F$ and $Y$, a new estranged pair of facets is created. Since this contradicts maximality, each such $Y$ must be a 3-gon and then it follows easily that $P$ is a pyramid over a 4-gon. $\square$

**2.5. Corollary.** *Among all 3-polytopes with a given number $f \geq 6$ of facets, the maximum number of inner diagonals is attained only by certain simple polytopes.*

*Proof.* This follows by polarity from Theorem 2.4. $\square$

We now proceed to determine both the minimum and the maximum number of inner diagonals for simple 3-polytopes with a given number $f \geq 6$ of facets. An essential notion is that of the *p-vector* $(p_3, p_4, \dots)$ of a 3-polytope $P$, where $p_i$ is the number of $i$-gonal facets of $P$. The $p$-vector $(p_3, p_4, \dots)$ is also written as $k_1^{p_{k_1}} \dots k_m^{p_{k_m}}$ with $3 \leq k_1 < k_2 < \cdots < k_m$, where $k_1, \dots, k_m$ are the values of $i$ for which $p_i > 0$. We occasionally deviate slightly from this practice to permit repetition among the $k_i$. For example, a wedge over an $(f-1)$-gon has $p$-vector $3^2 4^{f-4} (f-1)^2$ when $f \geq 6$, but when $f = 5$ the notation $3^2 4^1 4^2$ is to be interpreted as $3^2 4^3$.

For reasons that will immediately become apparent, we are interested in the function

$$\varphi(P) = \sum_i i^2 p_i,$$

where $(p_3, p_4, \dots)$ is the $p$-vector of $P$.

**2.6. Proposition.** *If a 3-polytope $P$ has $v(P)$ vertices and $f(P)$ facets, then*

$$\delta_2(P) = \frac{1}{2}\varphi(P) - 3\left(v(P) + f(P) - 2\right).$$

*Proof.* By double counting of incidences, $\sum_i i p_i = 2\delta_1$. Hence the conclusion follows from the fact that $\delta_1 = v(P) + f(P) - 2$ (Euler) in conjunction with the fact that the number of 2-diagonals of an $i$-gon is $\binom{i}{2} - i = (i^2 - 3i)/2$. $\square$

**2.7. Corollary.** *For each simple* 3*-polytope with* $f$ *facets,* $\delta_1 = 3f - 6$*,*

$$\delta_2 = \frac{1}{2}\varphi - 9f + 18$$

*and*

$$\delta_3 = 2f^2 - 3f - 2 - \frac{1}{2}\varphi.$$

*Proof.* Use Euler's theorem and double counting to see that $v = 2f - 4$ and $\delta_1 = 3f - 6$. Then apply Remark 1.1 and Proposition 2.6. □

It follows from Corollary 2.7 that if we are restricted to simple 3-polytopes with a given number $f$ of facets, the maximum (resp. minimum) number of inner diagonals is obtained by minimizing (resp. maximizing) the quantity $\varphi = \sum_i i^2 p_i$ over all sequences $(p_3, p_4, ...)$ that have $\sum_i p_i = f$ and are realized as the $p$-vector of some simple 3-polytope. Easy counting (using Euler's theorem and the fact that $2\delta_1 = 3v$) shows that the equation,

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{i \geq 7} (i-6)p_i$$

is a necessary condition for such realizability. The quantity $p_6$ does not appear in this equation, and a theorem of Eberhard [Eb] asserts that for each sequence satisfying the equation there is at least one value of $p_6$ for which the sequence with this $p_6$ in place is indeed the $p$-vector of a simple 3-polytope. (See Grünbaum [Gr1] for a much simpler proof.) However, for many choices of the reduced sequence $(p_i : 3 \leq i \neq 6)$ it is not known which values of $p_6$ have the stated property. For further information on this question, see Theorem 2.13 at the end of this section, and the references that precede it.

We turn now to the problem of minimizing $\delta_3$ over all simple 3-polytopes that have a given number $f$ of facets. This brings us back to the *wedges* mentioned briefly in Remark 1.3, for it will turn out that except for additional minimizers when $f$ is 7 or 8, the 3-dimensional wedges are precisely the desired minimizers. We define an $f$*-wedge* as a 3-polytope that is a wedge over an $(f-1)$-gon $K$ and has as its foot some edge of $K$. Each $f$-wedge is a simple 3-polytope that has $f$ facets, and for given $f$ all $f$-wedges are combinatorially equivalent. Figure 1 depicts a 6-wedge and its two inner diagonals.

**2.8. Lemma.** *An* $f$*-facet simple* 3*-polytope is an* $f$*-wedge if and only if its* $p$*-vector is* $3^2 4^{f-4}(f-1)^2$*. Each* $f$*-wedge* $W$ *has* $\varphi(W) = 2f^2 + 12f - 44$ *and* $\delta_3(W) = f^2 - 9f + 20$*.*

*Proof.* Simple calculation shows that if $W$ is an $f$-wedge then $W$'s $p$-vector and values for $\varphi$ and $\delta_3$ are as stated. To complete the proof we show that if $P$ is an

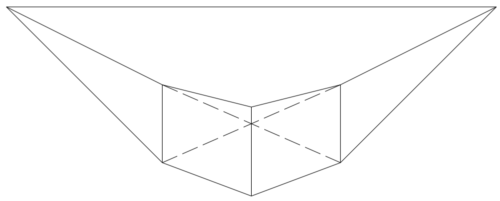

FIGURE 1. A 6-wedge and its two inner diagonals.

$f$-facet simple 3-polytope whose $p$-vector is $3^2 4^{f-4}(f-1)^2$, then $P$ is an $f$-wedge. Note that the two $(f-1)$-gonal facets $X$ and $Z$ of $P$ use a total of at least $2f-4$ vertices, and use more than this unless they have a common edge. But $P$, being simple, has only $2f-4$ vertices in all, so $X$ and $Z$ do have a common edge and, aside from the endpoints of this edge, the remaining $2f-6$ vertices are evenly divided between $X$ and $Z$. From this it is clear that the remaining edges of $P$ set up a matching between $f-3$ vertices of $X$ and $f-3$ vertices of $Z$, and because the graph of $P$ is planar, there is only one way of doing this. It follows that $P$ is an $f$-wedge. $\square$

It follows from an observation of Steinitz [SR] (see also [Gr1], p. 243) that for $f \geq 4$, each simple 3-polytope $P$ with $f+1$ facets can (in combinatorial type) be obtained from some simple 3-polytope $Q$ with $f$ facets by the sort of facet-splitting that we now describe.

Suppose that $Y$ is a facet of $Q$, $A = [a', a'']$ and $B = [b', b'']$ are distinct edges of $Y$, and $X$ and $Z$ are facets of $Q$ with $X \cap Y = A$ and $Z \cap Y = B$. Now add a new vertex $a_0$ in the open edge $]a', a''[$, a new vertex $b_0$ in the open edge $]b', b''[$, and a new edge $[a_0, b_0]$ in $Y$. The result is to replace the edge $A$ by the pair of edges $[a', a_0]$ and $[a_0, a'']$, to replace the edge $B$ by the pair of edges $[b', b_0]$ and $[b_0, b'']$, to replace the facet $Y$ by a pair of facets both incident to the new edge $[a_0, b_0]$, and to increase by 1 the number of edges of each of the facets $X$ and $Z$. Since this alteration of $Q$'s graph preserves planarity and 3-connectedness, the resulting graph must be isomorphic to that of some 3-polytope $P$. Note that $P$ is simple and $f(P) = f(Q) + 1$.

If the numbers of vertices of $X$, $Y$, and $Z$ are respectively $\alpha$, $\beta$, and $\gamma \leq \alpha$, and the $\beta$-gon $Y$ is split into a $\xi$-gon and an $\eta$-gon with $\eta \leq \xi$, then the above sort of transition from the simple 3-polytope $Q$ to the simple 3-polytope $P$ is here called an $(X, Y, Z)$-*splitting of type* $(\alpha : \xi, \eta : \gamma)$. Note that $\xi + \eta = \beta + 4 \geq 7$.

As can be seen in Figure 2, an $(f+1)$-wedge can be obtained from an $f$- wedge by means of splittings of three different types: $(f-1 : 4, 4 : f-1)$, $(f-1 : 4, 3 : f-1)$, and $(f-1 : f, 3 : 3)$. In each case the value of $\varphi$ increases by $4f + 14$.

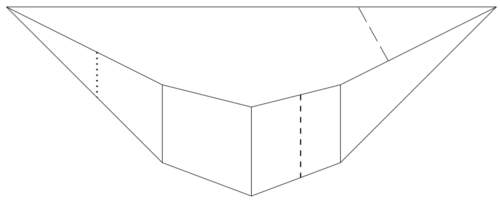

FIGURE 2. A 7-wedge can be obtained from a 6-wedge by three different types of splittings

**2.9. Lemma.** *Suppose that $Q$ is a simple 3-polytope with $f$ facets, and $P$ arises from $Q$ by means of a splitting of type $t = (\alpha : \xi, \eta : \gamma)$. Then*

$$\varphi(P) - \varphi(Q) \leq 4f + 14,$$

*with equality if and only if*

$$X \cap Y \cap Z = \emptyset \neq X \cap Z \quad \textit{and} \quad t = (f-1 : 4, 4 : f-1)$$

*or*

$$X \cap Y \cap Z \neq \emptyset, \quad \eta = 3, \quad \textit{and} \quad \alpha + \xi + \gamma = 2f + 2.$$

*In each case, equality implies that all vertices of $Q$ belong to $X \cup Y \cup Z$.*

*Proof.* Let

$$\psi(t) = \psi(\alpha : \xi, \eta : \gamma) = \alpha + \gamma + 4\xi + 4\eta - \xi\eta.$$

In the passage from the $p$-vector of $Q$ to that of $P$, an $\alpha$ is replaced by an $\alpha + 1$, a $\gamma$ is replaced by a $\gamma + 1$, and the $\beta$ associated with the split facet is replaced by a $\xi$ and an $\eta$ with $\xi + \eta = \beta + 4$. Hence

$$\varphi(P) - \varphi(Q) = (\alpha+1)^2 - \alpha^2 + (\gamma+1)^2 - \gamma^2 + (\xi^2 + \eta^2 - \beta^2) = 2\psi(t) - 14.$$

It follows that $\varphi(P) - \varphi(Q) \leq 4f + 14$ if and only if $\psi(t) \leq 2f + 14$, with equality in either case corresponding to that in the other. Thus we may focus on $\psi$.

It is obvious that the following constraints are satisfied by the type $(\alpha : \xi, \eta : \gamma)$ of any splitting of a simple 3-polytope $Q$ with $f$ facets:

$$3 \leq \gamma \leq \alpha \leq f - 1 \tag{1}$$

$$3 \leq \eta \leq \xi \leq f \tag{2}$$

From the fact that $Q$ is simple it follows also that

(3) $Q$ has exactly $2f-4$ vertices;

(4) $\eta = 3$ if and only if $X \cap Y \cap Z \neq \emptyset$;

(5) any two intersecting facets of $Q$ have exactly two common vertices

Let $u$ denote the total number of vertices used by $X \cup Y \cup Z$. With the aid of (5), elementary counting shows that $\alpha+\beta+\gamma = u+k$ where

$$k = \begin{cases} 4 & \text{when } X \cap Z = \emptyset; \\ 5 & \text{when } X \cap Y \cap Z \neq \emptyset; \\ 6 & \text{when } X \cap Y \cap Z = \emptyset \neq X \cap Z. \end{cases}$$

Thus

$$(6_k) \qquad \alpha+\gamma+\xi+\eta = u+4+k\,,$$

whence

$$(7_k) \qquad \psi(t) = u+4+k+3\xi+3\eta-\xi\eta$$

$$(8_k) \qquad \leq 2f+k+3\xi+3\eta-\xi\eta\,,$$

with equality if and only if $u = 2f-4$.

When $X \cap Y \cap Z = \emptyset$ (i.e. $k \in \{4,6\}$), we know from (2) and (4) that $4 \leq \eta \leq \xi$. Now let $\nu(\xi,\eta) = 3\xi+3\eta-\xi\eta$ and note that $\partial\nu/\partial\xi = 3-\eta$, $\partial\nu/\partial\eta = 3-\xi$. Hence in the open quarter-plane given by $\xi > 3$, $\eta > 3$, the function $\nu$ is strictly decreasing in $\xi$ for each fixed $\eta$ and strictly decreasing in $\eta$ for each fixed $\xi$. Applying this observation and $(8_k)$ it follows that

$$\begin{aligned} \psi(t) &\leq 2f+k+(3\times 4)+(3\times 4)-(4\times 4) \\ &= 2f+8+k. \end{aligned}$$

with equality if and only if $\xi = 4$, $\eta = 4$, and $u = 2f-4$. If $k = 4$ then the upper bound of $2f+14$ is not reached. If $k = 6$ then $(6_6)$ tells us that $\alpha+\gamma+8 = 2f+6$, so $\alpha+\gamma = 2f-2$ and it follows with the aid of (1) that $\alpha = f-1 = \gamma$. In other words, when $X \cap Y \cap Z = \emptyset$, $\psi$'s upper bound of $2f+14$ is attained if and only if $X \cap Z \neq \emptyset$, $u = 2f-4$, and $t = (f-1:4,4:f-1)$.

When $X \cap Y \cap Z \neq \emptyset$ we have $k = 5$ and $\eta = 3$, whence (by $(7_k)$),

$$\psi(t) = u+18 \leq 2f+14$$

($6_5$) now tells us that $\alpha + \gamma + \xi \le 2f + 2$, with equality if and only if $u = 2f - 4$. That completes the proof. □

We have seen in Lemma 2.8 that an $f$-wedge $P$ has $\varphi(P) = 2f^2 + 12f - 44$ and is characterized among simple 3-polytopes by having $3^2 4^{f-4}(f-1)^2$ as its $p$-vector. We have also seen (in Figure 2) that for $f \ge 6$ an $(f+1)$-wedge can be produced from an $f$-wedge by means of splittings of three different types: $(f-1:4, 4:f-1)$, $(f-1:4, 3:f-1)$ and $(f-1:f, 3:3)$.

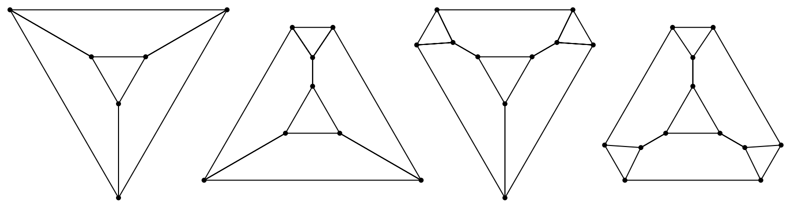

FIGURE 3. The polytopes $T_k$ obtained by truncating a tetrahedron at $k-4$ of its vertices.

For $5 \le f \le 8$, the $T_f$ of Figure 3 is a simple 3-polytope that has $f$ facets and arises from truncating a tetrahedron at $f-4$ of its vertices. In each case, $\varphi(T_f) = 2f^2 + 12f - 44$. The $p$-vectors for $T_5$, $T_6$, $T_7$, and $T_8$ are $3^2 4^3$, $3^2 4^2 5^2$, $3^3 5^3 6$, and $3^4 6^4$ respectively; note that $T_5$ and $T_6$ are wedges but $T_7$ and $T_8$ are not. The latter two play a special role in the following theorem.

**2.10. Theorem.** *If $f \ge 4$ and $P$ is a simple 3-polytope with $f$ facets, then*

$$\varphi(P) \le 2f^2 + 12f - 44 \quad \text{and} \quad \delta_3(P) \ge f^2 - 9f + 20.$$

*Equalities hold if and only if $P$ is an $f$-wedge or $f$ is 7 or 8 and $P$ is formed by truncating a tetrahedron at $f-4$ of its vertices.*

*Proof.* By Corollary 2.7, the statements about $\varphi$ and $\delta_3$ are equivalent. We will concentrate on $\varphi$ and apply Lemma 2.9, proceeding by induction on $f$. Our claim is that $T_7$ and $T_8$ are (in combinatorial type) the only $f$-facet simple 3-polytopes that are not wedges and yet have $\varphi = 2f^2 + 12f - 44$.

When $f$ is 4 or 5, there is only one combinatorial type of simple 3-polytope with $f$ facets. When $f$ is 6 there are just two combinatorial types: a cube with $p$-vector $4^6$ and $\varphi = 96$ and a 6-wedge with $p$-vector $3^2 4^2 5^2$ and $\varphi = 100$. (Both of these statements can be established by considering all possible splits of the tetrahedron and $T_5$ respectively.)

Now let $\mu(f)$ denote the maximum of $\varphi$ over all simple 3-polytopes with $f$ facets. It is evident, for each $f$, that $\mu(f+1)$ cannot exceed $\mu(f)$ by more than the maximum, over *all* simple 3-polytopes with $f$ facets, of the increase in $\varphi$ that is attainable by splitting a facet. By Lemma 2.9, this maximum increase is $4f+14$ and can be attained only when the splitting is of certain specified types and all vertices of the polytope belong to the union of three facets involved in the splitting.

Starting from a 6-wedge (e.g., $T_6$), Lemma 2.9 leads us to the three types of splitting (mentioned above) that result in a 7-wedge, plus two other $\psi$-maximizing types of splitting: $(5:5,3:4)$ or $(4:6,3:4)$. Both of the latter result in (a combinatorial equivalent of) $T_7$.

In the case of $T_7$ there are no 4-gonal facets, so the only possible type $t$ for a splitting that has $\psi(t)=2f+14$ is one in which $\eta=3$, whence the three involved facets have a common vertex $s$, and their union uses all 10 vertices. In $T_7$ there is only one choice for $s$, the splitting is of type $(5:6,3:5)$, and it produces $T_8$. The splitting situation for $T_8$ is similar to that for $T_7$, except that in $T_8$ there is *no* choice for $s$.

It remains to show that each of the $\varphi$-maximizing splits of Lemma 2.9 yields an $(f+1)$-wedge when applied to an $f$-wedge for $f\geq 7$. In the case of $(f-1:4,4:f-1)$ splits, this is clear, so we consider the second type of split described in the lemma. In this case there are two (symmetric) choices for the vertex $s=X\cap Y\cap Z$, and both resulting types of split yield wedges. Hence the list of non-wedge $\varphi$-maximizers consists of $T_7$ and $T_8$; for $f\geq 9$ the only maximizers are the wedges. $\square$

We turn now to the problem of maximizing the number of inner diagonals in the class of all simple 3-polytopes having a given number $f$ of facets. For $f\geq 14$ the maximizers turn out to be polytopes whose existence was established by Grünbaum and Motzkin [GM]. We start by establishing the non-existence of simple 3-polytopes with certain $p$-vectors.

**2.11. Remark.** *If a simple 3-polytope has $f$ facets and $f-1$ of them are pentagons, then $f=12$ and all facets are pentagons.*

*Proof.* Suppose that $F$ is a $k$-gonal facet of $P$ and all the remaining facets are pentagons. By counting the edges in two different ways, we see that $5(f-1)+k=2\delta_1=3v$, and since $v=2f-4$ it follows that $k=f-7$. Since each facet that intersects $F$ must share an edge with $F$, there are $k$ pentagons adjacent to $F$ and $f-(k+1)=6$ additional pentagons. The union of $F$ and its $k$ adjacent pentagons forms a graph in which there are precisely $k$ vertices of degree 2, and each of these must be covered by one of the additional pentagons. When these $k$ additional pentagons are added, we have a graph in which the outer boundary is a $k$-gon and each vertex is of degree 3. Since we are building up a simple 3-polytope, the buildup is now complete and there is just one facet (corresponding to the outer boundary) not yet accounted for. Thus $f = 2k + 2$, whence $f = 12$ and $k = 5$. □

The following characterization was suggested by the results of a computer search using an algorithm of Avis [Av].

**2.12. Theorem.** *For simple 3-polytopes with $f$ facets, the maximum number of inner diagonals and the unique associated p-vector are as follows:*

| $f$ | $\delta_3$ | *p-vector* |
|---|---|---|
| $f < 6$ | $0$ | |
| $6 \le f \le 10$ | $2f^2 - 20f + 52$ | $4^{12-f}5^{2f-12}$ |
| $11$ | $73$ | $4^2 5^8 6$ |
| $12$ | $100$ | $5^{12}$ |
| $13$ | $128$ | $4^1 5^{10} 6^2$ |
| $14 \le f$ | $2f^2 - 21f + 64$ | $5^{12} 6^{f-12}$ |

*Proof.* Let $P$ be a simple 3-polytope with $f$ facets. By Corollary 2.7,

$$\delta_3(P) = 2f^2 - 3f - 2 - \frac{1}{2}\varphi(P),$$

and hence an upper bound on the number of $P$'s inner diagonals can be obtained by solving the following linear program.

$$\begin{aligned} &\min \varphi \\ &\sum_{i\ge 3}(6-i)p_i = 12 \\ &\sum_{i\ge 3} p_i = f \\ &p_i \ge 0 \end{aligned} \tag{9}$$

As before, the first equation is a consequence of Euler's formula and the fact that $P$ is simple. Rearranging the equality constraints of (9) in two different ways, we obtain:

(10a) $p_4 = (12 - f) - \sum_{i\notin\{4,5\}} (5 - i)p_i \qquad p_5 = (2f - 12) + \sum_{i\notin\{4,5\}} (4 - i)p_i$

(10b) $p_5 = 12 - \sum_{i\notin\{5,6\}} (6 - i)p_i \qquad p_6 = (f - 12) + \sum_{i\notin\{5,6\}} (5 - i)p_i$

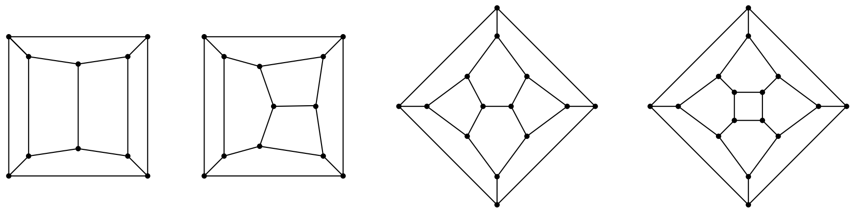

FIGURE 4. Simple $f$-facet 3-polytopes with the maximum number of inner diagonals for $7 \leq f \leq 10$.

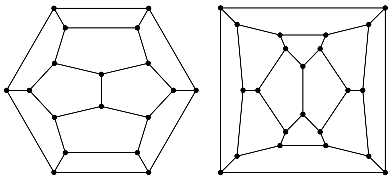

FIGURE 5. Simple $f$-facet 3-polytopes with the maximum number of inner diagonals for $f = 11$ and $f = 13$.

Observe that the $p$-vector $4^{12-f}5^{2f-12}$ (resp. $5^{12}6^{f-12}$) is feasible for (9) if $f \leq 12$ (resp. if $f \geq 12$). We now argue that these two classes of solutions are optimal. Substituting (10a) and (10b) respectively into $\varphi$ yields

$$\varphi_a = 34f - 108 + \sum_{i \notin \{4,5\}} (4-i)(5-i)p_i\,,$$
$$\varphi_b = 36f - 132 + \sum_{i \notin \{5,6\}} (5-i)(6-i)p_i\,.$$

The objective function $\varphi_a$ (resp. $\varphi_b$) is minimized uniquely (over nonnegative $p_i$) by setting $p_i = 0$, $i \notin \{4,5\}$ (resp. $i \notin \{5,6\}$).

It turns out that there exist simple 3-polytopes corresponding to all but two of these minimizing $p$-vectors. The cube and the dodecahedron provide examples for $f = 6$ and $f = 12$ respectively. Figure 4 shows examples for $7 \leq f \leq 10$. Grünbaum and Motzkin [GM] showed that for each $f \geq 14$ there are simple 3-polytopes with the $p$-vector $5^{12}6^{f-12}$.

It remains to consider the cases of $f = 11$ and $f = 13$. By Remark 2.11, neither of the corresponding minimizing $p$-vectors for (9) is realized by a simple 3-polytope. Since the minimizing $p$-vectors for (9) are unique and the value of $\varphi$ must be an even integer, we conclude that the smallest value of $\varphi$ that corresponds to a *realizable* $p$-vector is at least two more than the minimum obtained from (9) and hence the corresponding upper bound for the number of inner diagonals is reduced by at least one. These reduced upper bounds on $\delta_3$ are sharp for $f \in \{11, 13\}$, as is shown by the polytopes depicted in Figure 5. We can write the corresponding minimal values for the objective $\varphi$ in the form

$$\sum_{i \geq 3} i^2 p_i = 35f - 117. \tag{11}$$

Solving the equality constraints of (9), along with (11), for $p_4$, $p_5$, and $p_6$, we obtain

$$p_4 = (15 - f)/2 \; -3p_3 - \frac{1}{2}\sum_{i \geq 7} (i-5)(i-6)p_i, \tag{12a}$$

$$p_5 = (f - 3) \qquad +3p_3 \;\; + \sum_{i \geq 7} (i-4)(i-6)p_i, \tag{12b}$$

$$p_6 = (f - 9)/2 \quad - p_3 - \frac{1}{2}\sum_{i \geq 7} (i-4)(i-5)p_i. \tag{12c}$$

Since for each $p_i$ we have $0 \leq p_i \leq f$, equation (12a) implies $p_3 = 0$. For $i \geq 7$, $(i-4)(i-5)/2 \geq 3$, hence (12c) implies $p_i = 0$ for $i \geq 7$. It follows that here again there is a unique choice of realizable $p$-vectors that maximize the number of inner diagonals. □

To facilitate comparisons, we summarize the results of Theorems 2.2, 2.10, and 2.12 in terms of both vertices and facets.

- The number of inner diagonals of each simplicial 3-polytope with $v$ vertices is $(v^2 - 7v + 12)/2$, while for simple 3-polytopes with $v$ vertices the number ranges from $(v^2 - 10v + 24)/4$ to (when $v \geq 24$) $(v^2 - 13v + 60)/2$.
- The number of inner diagonals of each simplicial 3-polytope with $f$ facets is $(f^2 - 6f + 8)/8$, while for simple 3-polytopes with $f$ facets the number ranges from $f^2 - 9f + 20$ to (when $f \geq 14$) $2f^2 - 21f + 64$.

For each $f \geq 6$, we have determined both the minimum number and the maximum number of inner diagonals among all simple 3-polytopes with $f$ facets. There remains the problem of deciding which intermediate values for $\delta_3$ can be attained for a given $f$. We have not attacked this problem, but it seems probable that the following result of Jendrol' [Je] (extending earlier work of Grünbaum [Gr1], [Gr2], Jucovič [Ju], and Fisher [Fi]) can be used to settle the problem for all sufficiently large $f$ (see also Roudneff [Rou]).

**2.13. Theorem (Jendrol').** *Suppose that the sequence $p = (p_i : 3 \leq i \neq 6)$ is such that*

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{i \geq 7} (i-6)p_i,$$

*and let $P(p)$ denote the set of all integers $p_6 \geq 0$ for which the sequence $(p_i : 3 \leq i)$ is the full p-vector of some simple 3-polytope. Let*

$$\sigma = \sum_{3 \leq i \neq 6} p_i, \qquad \text{and} \qquad \tau = \sum_{3 < j \not\equiv 0 \pmod 3} p_j.$$

(a) *If $\tau \leq 2$ and $\sigma$ is even, then $P(p)$ includes no odd integer but includes all sufficiently large even integers.*
(b) *If $\tau \leq 2$ and $\sigma$ is odd, then $P(p)$ includes no even integer but includes all sufficiently large odd integers.*
(c) *If $\tau \geq 3$ then $P(p)$ includes all sufficiently large integers.*

## 3. Inner diagonals of higher-dimensional polytopes

A polytope is called 2-*neighborly* if each pair of its vertices defines an edge.

**3.1. Proposition.** *For each $n > d \geq 4$, there exists a simplicial $d$-polytope that has $n$ vertices and no inner diagonal and there also exists a simple $d$-polytope that has $n$ facets and no inner diagonal.*

*Proof.* For the first assertion, consider the cyclic $d$-polytopes with $n$ vertices [Ca], [Ga]. They are simplicial and when $d \geq 4$ they are 2-neighborly.

For the second assertion, note that if $P$ is the sort of 3-polytope that we have called an $f$-wedge and $F$ is either of the $(f-1)$-gonal facets of $P$, then each inner diagonal of $P$ intersects $F$. From this fact in conjunction with Remark 1.3 it follows that if $W$ is a wedge over $P$ with foot $F$, then $W$ is a simple 4-polytope with $f+1$ facets and no inner diagonal. From there on, repeated wedging (over an arbitrary facet) produces higher-dimensional simple polytopes that have no inner diagonal. □

Let $X$ be the vertex set of a $d$-polytope $P \subset \mathbb{R}^d$, and $x_0 \in X$. Let $X' = (X \setminus \{x_0\}) \cup \{x_0'\}$, where $x_0'$ is a point of $\mathbb{R}^d$ such that the half-open segment $]x_0, x_0']$ does not intersect any hyperplane determined by points of $X$. If $x_0$ is in the interior of $P'$, $P' = \operatorname{conv} X'$ is said to be *obtained from $P$ by pulling $x_0$*. If $x_0'$ is in the interior of $P$, $P'$ is said to be *obtained from $P$ by pushing $x_0$*. Initially we will consider some consequences of pulling; we return to pushing below.

It is shown in [EGK] and [Gr1] that if $\operatorname{conv} X'$ is obtained from $\operatorname{conv} X$ by pulling, then for $0 \le r \le d-1$, the $r$-faces of $\operatorname{conv} X'$ are precisely the sets of the following two sorts: an $r$-face of $\operatorname{conv} X$ that misses $x_0$; a pyramid of the form $\operatorname{conv}(B \cup \{x_0'\})$, where $B$ is an $(r-1)$-face of a facet $F$ of $\operatorname{conv} X$ such that $x_0 \in F \setminus B$.

Let $\rho(s,t,P)$ denote the dimension of the carrier of $[s,t]$ in polytope $P$.

**3.2. Lemma.** *Suppose that $x_0$ belongs to the vertex set $X$ of a $d$-polytope $P \subset \mathbb{R}^d$, and that $P'$ is obtained from $P$ by pulling $x_0$ to a new position $x_0'$. Then the following statements are true.*

(a) *The number of estranged facet-pairs in $P'$ is not less than the number in $P$.*
(b) *If $P$ has an estranged facet-pair $(F,G)$ such that $x_0 \in F$ and $F$ is not a pyramid with apex $x_0$, then $P'$ has more estranged facet-pairs than $P$.*
(c) *If a vertex-pair $(x,y)$ is estranged in $P$, then the pair $(x',y')$ is estranged in $P'$. Hence the number of inner diagonals in $P'$ is not less than the number in $P$.*
(d) *With $1 < k < d$, suppose that $[s,t]$ is a $k$-diagonal of $P$, that $K$ is the carrier of $[s,t]$ in $P$, and that $x_0$ is a vertex of $K$ that is neither $s$ nor $t$. Then $\rho(s,t,P') > \rho(s,t,P)$.*

*Proof.* Let $x' = x$ for each $x \in X \setminus \{x_0\}$, and set $X' = \{x' : x \in X\}$. (Thus $P' = \operatorname{conv} X'$.) For (a) and (c), note that each facet of $P$ that misses $x_0$ is not affected by the pulling, while each facet $F$ of $P$ that is incident to $x_0$ is (in combinatorial effect) replaced by a subdivision of $F$ that consists of pyramids with common apex $x_0$. For (b), note further that if $F$ is not already a pyramid with apex $x_0$, then the subdivision is proper, the combinatorial effect being to replace the single facet $F$ by two or more $(d-1)$-pyramids, each of which yields (after the pulling) a facet of $P'$.

For (d) it suffices to show, under the stated conditions, that if $K'$ is a $k$-face of $P'$ then $K'$ does not contain the segment $[s,t]$. Indeed, if $x_0' \notin K'$ then $K'$ is a face of $P$ and it cannot contain $[s,t]$ because $K$ (being the carrier of $[s,t]$) is the unique $k$-face of $P$ that contains $[s,t]$. On the other hand, if $x_0' \in K'$ then $K'$ is a $k$-dimensional pyramid with apex $x_0'$, from which it follows that if $[s,t] \subset K'$ then $[s,t]$ lies in the base $B$ of the pyramid, contradicting the assumption that the carrier of $[s,t]$ in $P$ is of dimension $k$. $\square$

**3.3. Corollary.** *If $d \ge 2$ and $P$ is a $d$-polytope whose number of inner diagonals is a maximum among all $d$-polytopes having a given number $v$ of vertices, then $\delta_1(P) + \delta_d(P) = \binom{v}{2}$. Equivalently, each facet of $P$ is 2-neighborly.*

*Proof.* Let $\mathcal{P}$ denote the class of all $v$-vertex $d$-polytopes that maximize the number of inner diagonals. For each $P \in \mathcal{P}$, define the integer $\tau(P)$ by the following two properties:

(i) $1 \leq \tau(P) < d$ and $\delta_{\tau(P)}(P) > 0$;
(ii) $\delta_r(P) = 0$ for each $r$ with $\tau(P) < r < d$.

The corollary asserts that $\tau(P) = 1$ for each $P \in \mathcal{P}$.

Suppose the assertion is false, and consider a member $P$ of $\mathcal{P}$ for which $\tau(P)$ is maximized. Then $1 < \tau(P) < d$. An application of 3.2(d) then yields, when $\tau(P) = d - 1$, a $v$-vertex $d$-polytope $P'$ such that $\delta_d(P') > \delta_d(P)$, and when $\tau(P) < d - 1$ it yields a member $P'$ of $\mathcal{P}$ such that $\tau(P') > \tau(P)$. In each case a maximizing property of $P$ is contradicted. $\square$

We now argue that the $v$-vertex $d$-polytopes maximizing the number of inner diagonals are all simplicial. Let $p_k(P)$ denote the number of 2-faces of $P$ with $k$-vertices. For a $v$-vertex $d$-polytope $P$, define the functions $g_1(P)$ and $g_2(P)$ as follows:

$$\begin{aligned} g_1(P) &= v - (d+1) \\ g_2(P) &= \delta_1(P) - \left[dv - \binom{d+1}{2}\right] + \sum_{k>3}(k-3)p_k(P)\,. \end{aligned}$$

A $v$-vertex stacked $d$-polytope is either a $d$-simplex or is obtained recursively from a $(v-1)$-vertex stacked polytope by erecting a pyramid over one of the facets. Barnette's Lower Bound Theorem [Ba] says that for each $j$ with $1 \leq j \leq d-1$, each $v$-vertex simplicial $d$-polytope has at least as many $j$-faces as a $v$-vertex stacked $d$-polytope, and that within the class of simplicial polytopes, this bound is attained (when $d \geq 4$) only by the stacked polytopes. The Lower Bound Theorem can be recursively reduced to the case $j = 1$ (see [Ka1, Sec. 5] for details); hence for simplicial $d$-polytopes with $d \geq 4$ the Lower Bound Theorem is equivalent to the statements

(L1) $\qquad g_2(P) \geq 0, \qquad$ and

(L2) $\qquad g_2(P) = 0 \qquad$ if and only if $P$ is a stacked polytope.

The statement (L1) was proved for rational (not necessarily simplicial) polytopes by Stanley [St]. Kalai [Ka1] generalized this to all polytopes by using notions of rigidity of graphs, in particular a theorem of Whitely [Wh].

**3.4. Theorem (Kalai).** *For each polytope $P$, $g_2(P) \geq 0$.*

In the same paper, Kalai generalized the Lower Bound Theorem (and hence in particular (L2)) to triangulated $(d-1)$-manifolds. (Note that the given definition of $g_2$ applies to a simplicial complex as well.)

**3.5. Theorem (Kalai).** *If $\Delta$ is a triangulated $(d-1)$-manifold with $d \geq 4$, then*

(a) $g_2(\Delta) \geq 0$*, and*
(b) $g_2(\Delta) = 0$ *if and only if $\Delta$ is isomorphic to the boundary complex of a stacked $d$-polytope.*

Note that part (a) of 3.5 is neither directly implied by nor directly implies 3.4. As an immediate consequence of Theorem 3.5, we have

**3.6. Proposition.** *Among the triangulated $(d-1)$-manifolds with $v$ vertices, the maximum number of estranged vertex-pairs (i.e. pairs not contained in any $(d-1)$-simplex) is $\binom{v}{2} - dv + \binom{d+1}{2}$, and this bound is attained by precisely the triangulated manifolds that are isomorphic to the boundary complex of a stacked polytope.*

In order to generalize statement (L2) to not necessarily simplicial polytopes, we need the following theorem of Kalai [Ka2]. Let $P/F$ denote the quotient polytope of $P$ with respect to $F$, i.e., a polytope whose face lattice is isomorphic to the interval $\{\, G \mid F \subseteq G \subseteq P\}$ of the face lattice of $P$ (see [Zi], p. 57).

**3.7. Theorem (Kalai).** *For any $d$-polytope $P$ and any face $F$ of $P$,*

$$g_2(P) \geq g_2(F) + g_1(F)g_1(P/F) + g_2(P/F)\,.$$

**3.8. Proposition.** *For $d \geq 4$, if $P$ is a $d$-polytope with 2-neighborly facets, then $g_2(P) = 0$ if and only if $P$ is a stacked polytope.*

*Proof.* Let $P$ be a $v$-vertex $d$-polytope with 2-neighborly facets such that $g_2(P) = 0$. Suppose that $P$ is not simplicial, i.e., there exists some facet $F$ of $P$ with $k > d$ vertices.

$$g_2(F) = \binom{k}{2} - (d-1)k + \binom{d}{2} = (k-d)(k-d+1)/2\,.$$

By Theorem 3.7,

$$g_2(P) \geq (k-d)(k-d+1)/2 + (k-d)0 + g_2(P/F)\,.$$

By Theorem 3.4, $g_2(P/F) \geq 0$, thus $g_2(P) > 0$. Since this is a contradiction, $P$ must be simplicial. The proposition then follows from the Lower Bound Theorem for simplicial polytopes. □

**3.9. Theorem.** *Within the class of $d$-polytopes that have a given number $v$ of vertices, the maximum possible number of inner diagonals is $\binom{v}{2} - dv + \binom{d+1}{2}$. For $d \geq 4$ this maximum is attained by the stacked polytopes and only by them.*

*Proof.* Let $P$ be a $v$-vertex $d$-polytope that maximizes the number of inner diagonals. By Corollary 3.3, every 2-face of $P$ is a triangle and $\delta_d(P) = \binom{d}{2} - \delta_1(P)$. By Theorem 3.4, $\delta_1(P) \geq dv - \binom{d+1}{2}$. By Proposition 3.8, for $d \geq 4$, this minimum (and hence the maximum for $\delta_d$) is obtained only by the stacked $d$-polytopes. □

We now turn our attention to maximizing the number of inner diagonals among polytopes with a fixed number of facets.

**3.10. Theorem.** *Within the class of $d$-polytopes that have a given number $f \geq 2d$ of facets, the maximum possible number of inner diagonals is attained by certain simple $d$-polytopes.*

*Proof.* By Lemma 3.2, pulling a vertex does not decrease the number of estranged facet-pairs. After all vertices have been pulled, the resulting polytope must be simplicial, since each facet is pyramidal from each of its vertices. It follows that for each $d \geq 2$ and $v \geq 2d$ there are simplicial polytopes among the $v$-vertex $d$-polytopes that maximize the number of estranged facet-pairs. Theorem 3.10 is the polar equivalent of this statement. □

For $d \geq 4$ we do not know whether "by certain" in Theorem 3.10 can be replaced by "only by certain." We can however make a few observations about the $d$-polytopes that maximize the number of inner diagonals among those with a fixed number of facets. The following two statements are equivalent by polarity, and follow from part (b) of Lemma 3.2.

**3.11. Proposition.**

(a) *If $P$ is a $d$-polytope which, subject to the restriction of having a specified number of vertices, maximizes the number of estranged pairs of facets, then each facet of $P$ that is not a simplex must intersect every other facet of $P$.*

(b) *If $P$ is a $d$-polytope which, subject to the restriction of having a specified number of facets, maximizes the number of inner diagonals, then each vertex of $P$ whose degree exceeds $d$ must share a facet with every other vertex of $P$.*

Parts (a), (b) and (c) of the following lemma are from [Kl]. Part (d) can be established by applying Grünbaum's "beneath and beyond" analysis of the inductive construction of polytopes (p. 78 of [Gr1]).

**3.12. Lemma.** *Suppose that $x_0$ belongs to the vertex set $X$ of a $d$-polytope $P \subset \mathbb{R}^d$, and that $P'$ is obtained from $P$ by pushing $x_0$ to a new position $x_0'$. Let $Y$ be the vertex set of an $r$-face of $P$.*

(a) *If $x_0 \notin Y$, $\operatorname{conv} Y$ is an $r$-face of $P'$.*
(b) *If $x_0 \in Y$ and $Y$ is pyramidal at $x_0$, $\operatorname{conv}(Y \setminus \{x_0\} \cup \{x_0'\})$ is an $r$-face of $P'$.*
(c) *If $x_0 \in Y$ and $Y$ is not pyramidal at $x_0$, then*
  (i) *$F' = \operatorname{conv}(Y \setminus \{x_0\})$ is an $r$-face of $P'$, and*
  (ii) *For each $(r-1)$-face $G$ of $F'$ such that $G$ is not an $(r-1)$-face of $P$, $\operatorname{conv}(G \cup \{x_0'\})$ is an $r$-face of $P'$.*
(d) *Each face of $P'$ is obtained in exactly one of these ways.*

Let $\operatorname{dist}(s, t, P)$ (the *distance* from $s$ to $t$) denote the length (number of edges) of the shortest path from $s$ to $t$ in the graph defined by vertices and edges of polytope $P$. The *diameter* of a polytope $P$ with vertex set $X$ is defined as the maximum over all $\{x, y\} \subset X$ of $\operatorname{dist}(x, y, P)$.

**3.13. Theorem.** *If $P$ is a $d$-polytope whose number of estranged facet-pairs is a maximum among all $d$-polytopes having a given number $v$ of vertices, then each facet of $P$ has diameter at most 2.*

*Proof.* Suppose that $P$ maximizes, among all $v$-vertex $d$-polytopes, the number of pairs of estranged facets. Further suppose (for the purposes of obtaining a contradiction) that there is some facet $F = \operatorname{conv} Y$ of $P$ of diameter greater than 2. Let $s$ and $t$ be two vertices of $F$ at distance at least 3. Let $Q$ be the polytope obtained by successively pushing $s$ to $s'$ and $t$ to $t'$.

Since the apex of a pyramid is adjacent to every other vertex, $F$ is not pyramidal at $s$ and $\operatorname{conv}(Y \setminus \{s\})$ is not pyramidal at $t$. It follows by 3.12(c)(ii) that $Q$ has facets $F_s = \operatorname{conv}(\{s'\} \cup G_s)$ and $F_t = \operatorname{conv}(\{t'\} \cup G_t)$ where $(G_s \cup G_t) \subset F$. Let $\mathcal{N}(x, P)$ (the *neighbors* of $x$ in $P$) denote the vertices $y$ of $P$ such that $[x, y]$ is an edge of $P$. Suppose $F_s$ intersects $F_t$. Facets $F_s$ and $F_t$ are pyramidal at $s'$ at $t'$ respectively, so if $F_s$ intersects $F_t$, then $\operatorname{dist}(s', t', Q) = 2$, i.e., $s'$ and $t'$ have some mutual neighbor $z$ in $Q$. We claim the following:

$$z \in F, \tag{13}$$

$$\mathcal{N}(s', Q) = \mathcal{N}(s, P), \quad \text{and} \tag{14}$$

$$\mathcal{N}(t', Q) = \mathcal{N}(t, P). \tag{15}$$

For (13), note that $z \in F_s \cap F_t = G_s \cap G_t$. For (14), apply 3.12(b) and 3.12(d). For (15), use these along with the fact that $s$ and $t$ do not have a mutual neighbor in

$F$, so pushing $s$ does not disturb the neighbors of $t$. From (13), (14), and (15) it follows that $z$ is a mutual neighbor of $s$ and $t$ in $F$. Since this is a contradiction, $F_s$ and $F_t$ are estranged.

As in the case of pulling, the combinatorial effect of pushing is to replace each facet with a (possibly trivial) subdivision of itself. Each estranged facet-pair of $P$ is thus preserved (i.e. either present or subdivided) in $Q$. Since $F_s$ and $F_t$ do not arise by subdivision of an estranged facet-pair of $P$ (in fact they originate from the same facet), $Q$ has strictly more estranged facet-pairs than $P$, but the same number of vertices. □

**3.14. Corollary.** *If $P$ is a $d$-polytope whose number of estranged facet-pairs is a maximum among all $d$-polytopes having a given number $v$ of vertices, then $P$ is simplicial or $P$ has diameter at most* 4.

*Proof.* Suppose $P$ is not simplicial. Consider a pair of vertices $s$ and $t$ of $P$. There are two cases.

(i) $s$ and $t$ are contained only in simplicial facets.
(ii) one of $s$ and $t$ is contained in a non-simplicial facet.

We argue that in each case $\mathrm{dist}(s, t, P) \leq 4$. Let $U$ be a non-simplicial facet of $P$. By Proposition 3.11, $U$ shares a vertex with every facet. In case (i) it takes exactly one edge to reach $U$ from $s$ and $t$, and then (by 3.13) at most two edges in $U$ to complete the path. In case (ii) suppose without loss of generality $t \in U$. It then takes at most two edges to reach $U$ from $s$, and at most two edges to complete the path in $U$. □

In contrast to the situation (Theorem 3.3) when the number of vertices is fixed, we do not know (for $d \geq 4$) what is the maximum number of inner diagonals when the number of facets is fixed. The present note was motivated by the following attractive conjecture of von Stengel [vS], which arose from his study of the Nash equilibrium points of bimatrix games.

**3.15. Conjecture (von Stengel).** *Within the class of all simple $d$-polytopes having $2d$ facets, the maximum number of inner diagonals is the $2^{d-1}$ attained by a $d$-cube.*

There are two relevant 3-polytopes: the 3-cube with 4 inner diagonals and the 5-wedge with 2. Grünbaum and Sreedharan [GS] provided (in dual form) a catalog of the 37 different combinatorial types of simple 4-polytopes with 8 facets. Of these, six have no inner diagonals and only the 4-cube has as many as 8. Hence Conjecture 3.15 holds (even with "attained" replaced by "attained only") for $d \leq 4$. However, the conjecture is open for all $d \geq 5$.

Going beyond Conjecture 3.15, it is of interest, for each pair $(d, f)$ with $f > d \geq 3$, to determine the maximum possible number of inner diagonals for simple $d$-polytopes with $f$ facets. This problem is settled in Section 2 for $d = 3$, but the results of Section 2 do not appear to contribute toward settling 3.15. (Von Stengel's main interest in [vS] was a stronger conjecture that involves pairs of simple $d$-polytopes. This too is known to be correct for $d \leq 4$ [Ke], [MP], and is open for $d = 5$. However, [vS] used polars of cyclic polytopes to provide counterexamples for all $d \geq 6$, even though they do not provide counterexamples to 3.15.

Inner diagonals were also of interest to Mani [Ma], who found a formula for $k(d)$, defined as the minimum of $v(P)$ in the class of all $d$-polytopes $P$ such that each vertex of $P$ is on at least one inner diagonal. It follows from his results that $k(d) = 2d$ when $d \leq 7$ (with cross-polytopes as the only extrema), that $k(d) < 2d$ when $d \geq 8$, and that $k(d)$ is about $d + 2\sqrt{d}$ for large $d$. For each $d$ there are extrema that are simplicial, but for large $d$ it is not known whether all extrema are simplicial. (See [Ros] for a simpler version of one of Mani's proofs.)

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