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# POINT VISIBILITY GRAPHS AND $\mathcal{O}$-CONVEX COVER*

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ABSTRACT

A visibility relation can be viewed as a graph: the uncountable graph of a visibility relationship between points in a polygon $P$ is called the *point visibility graph* (PVG) of $P$. In this paper we explore the use of *perfect graphs* to characterize tractable subproblems of visibility problems. Our main result is a characterization of which polygons are guaranteed to have weakly triangulated PVGs, under a generalized notion of visibility called *$\mathcal{O}$-visibility*. Let $\mathcal{O}$ denote a set of line orientations. A connected point set $P$ is called $\mathcal{O}$-convex if the intersection of $P$ with any line with orientation in $\mathcal{O}$ is connected. Two points in a polygon are said to be $\mathcal{O}$-visible if there is an $\mathcal{O}$-convex path between them inside the polygon. Let $\mathcal{O}^{\perp}$ denote the set of orientations perpendicular to orientations in $\mathcal{O}$. Let $\mathcal{O}' \subseteq \mathcal{O}^{\perp}$ be the set of orientations $\theta$ such that a "reflex" local maximum in the boundary of $P$ exists with respect to $\theta$. Our characterization of which polygons have weakly-triangulated PVGs is based on restricting the cardinality and span of $\mathcal{O}'$. This characterization allows us to exhibit a class of polygons admitting a polynomial algorithm for $\mathcal{O}$-convex cover.

*Keywords:* convex cover, approximation, restricted orientation, generalized visibility

## 1. Introduction

Define the orientation of a line as the smaller of the two angles that it makes with the positive $x$-axis. A connected point set $P \subset \mathbb{R}^2$ is called *$\mathcal{O}$-convex* if the intersection of $P$ with any line with orientation in the set $\mathcal{O}$ is connected. Whether or not a given point set is $\mathcal{O}$-convex depends on the choice of the parameter $\mathcal{O}$.

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If $\mathcal{O} = \emptyset$ then any connected set is trivially $\mathcal{O}$-convex. If $\mathcal{O} = \{\theta\}$, then the $\mathcal{O}$-convex sets are the (connected) $(\theta + 90°)$-monotone ones. If $\mathcal{O} = \{0°, 90°\}$ then $\mathcal{O}$-convexity corresponds to the notion of *orthogonal convexity* studied by several previous authors.[2,4,11,14] Finally, if $\mathcal{O}$ is the set of all orientations, $\mathcal{O}$-convexity is simply the classical notion of convexity.

Two points in a polygon are called *visible* if there is a line segment between them that does not intersect the exterior of the polygon. Visibility and convexity are intimately connected—a polygon is convex if and only if every pair of points is visible. Two points in a polygon are called $\mathcal{O}$-visible if there is an $\mathcal{O}$-convex path between them that does not intersect the exterior of the polygon. It turns out that a polygon is $\mathcal{O}$-convex if and only if every pair of points is $\mathcal{O}$-visible (see e.g. [18]).

The problem of convex cover asks for the smallest cardinality set of convex polygons whose union is a given simple polygon. Other than its inherent geometric interest, the main motivation for studying this problem is applications in pattern recognition, where the goal is to find a canonical representation of a polygon as the union of "simpler" polygons.[21] Since convex cover is NP-Hard,[1,5,22] various authors have considered cover by "approximately convex" polygons.[10,24] $\mathcal{O}$-convexity approximates the standard notion of convexity in the sense that each convex polygon is the limit of a sequence of $\mathcal{O}$-convex polygons with increasingly large sets of orientations $\mathcal{O}$. In this paper we study the complexity of $\mathcal{O}$-convex cover, i.e. finding a minimum cardinality set of $\mathcal{O}$-convex polygons whose union is a given simple polygon.

Earlier work on $\mathcal{O}$-convexity was concerned with the *isothetic* or *orthogonal* case of $\mathcal{O} = \{0°, 90°\}$.[2,4,11,14] A typical motivation for this work is VLSI routing, where wiring using only horizontal and vertical segments is called *Manhattan routing*. In the last ten years it has become increasingly common for VLSI routers (and fabricators) to use four or more distinct wiring orientations (see e.g. [13]). In the VLSI literature, Sarrafzadeh and Wong[19,20] have proposed a program of generalizing known heuristics and algorithms for the orthogonal case to larger sets of orientations. In the computational geometry literature, Rawlins and Wood[17,18] generalized the framework of orthogonal visibility/convexity to arbitrary (not necessarily finite sets of orientations). This paper can be seen a part of a (theoretical) program of extending our knowledge of the orthogonal case to a more general setting.

Motwani, Raghunathan, and Saran[14] showed that for certain restricted classes of polygons, the minimum number of $\{0°, 90°\}$-convex polygons to cover a polygon can be found in polynomial time. In this paper we consider for how general a value of the parameter $\mathcal{O}$ the polynomial upper bounds of Motwani et al. can be extended.

Shermer[23] introduced the notion of a *point visibility graph* as a unifying combinatorial framework for visibility problems. The point visibility graph (PVG) of a polygon $P$ denotes the uncountable graph $G = (V, E)$ whose vertices are the points of $P$ and whose edges are exactly those pairs $(x, y)$ for which $x$ is visible to $y$ in $P$. In this paper we consider the $\mathcal{O}$-PVG of a polygon, where the edges are the $\mathcal{O}$-visible pairs of points in $P$. Where $\mathcal{O}$ may be understood from context, we abbreviate $\mathcal{O}$-PVG to PVG.

Our goal here is to use the graph theoretic framework of PVGs to characterize tractable subproblems of $\mathcal{O}$-convex cover. In order for the framework of PVGs to provide polynomial upper bounds for $\mathcal{O}$-convex cover, we need two things: a polynomial sized representation of the PVG, and a polynomial algorithm to solve the resulting graph theoretic problem. We concern ourselves with the first in Section 2. For the second, we rely on the theory of *perfect graphs.* For a graph $G$, $\alpha(G)$ denotes the size of the maximum independent set in $G$ and $k(G)$ the minimum number of cliques to cover $G$. A graph $G$ is called perfect if for any induced subgraph $H$, $\alpha(H) = k(H)$ (for an introduction to the theory of perfect graphs, see [7]). Grötschel, Lovász and Schrijver[8] showed how to use the ellipsoid method to solve several optimization problems (including clique cover and maximum independent set) in polynomial time on perfect graphs. More efficient algorithms are available for several subclasses of perfect graphs. One such subclass is the class of *weakly triangulated* graphs introduced by Hayward.[9] A graph is called weakly triangulated if neither it nor its graph theoretic complement contains an induced chordless $k$-cycle for $k \geq 5$. In the rest of this paper we abbreviate induced chordless $k$-cycle to $k$-hole. The graph theoretic complement of a $k$-hole will be called a $k$-antihole.

Many (straight line) visibility properties of a polygon depend only on the reflex vertices of the polygon. An edge of a polygon is called reflex if both of its endpoints are reflex. Culberson and Reckhow[4] call such edges of orthogonal polygons "dents" and observed that they play an important role in orthogonal visibility. Here we generalize the notion of dent to a reflex vertex or edge that is tangent to a line with orientation in $\mathcal{O}$. The orientation of a dent is defined as the normal to the tangent line that faces into the polygon. We call the set of orientations for which dents exist with respect to some fixed $\mathcal{O}$ the set of *dent orientations* (with respect to $\mathcal{O}$).

In this paper we shall give a characterization of when the $\mathcal{O}$-PVG of a polygon is weakly triangulated. This characterization will be based on the set of dent orientations in $P$. Although the previous results in the context of orthogonal visibility are stated only in terms of the number of dent orientations, there are restrictions on the geometry implied by the fact that $|\mathcal{O}| = 2$. Here we explicitly characterize these geometric restrictions, and extend the previous characterization to general $\mathcal{O}$-visibility. A set of orientations is called *cyclic* if it leaves no closed half circle of orientations empty, *acyclic* otherwise. The main result of this paper is that the $\mathcal{O}$-PVG of a polygon is weakly triangulated if it has less than three dent orientations (w.r.t. $\mathcal{O}$), or it has a cyclic set of three dent orientations (w.r.t. $\mathcal{O}$).

## 2. Dent Decompositions

Although point visibility graphs provide a combinatorial framework for many visibility problems, in order to be computationally useful, a finite representation of the PVG is needed. In this section we describe such a representation (for the case where $\mathcal{O}$ is finite), based on decomposing the polygon into regions that see the same set of points. We start by observing a useful connection between $\mathcal{O}$-convexity and monotonicity. Let $\vec{u}$ and $\vec{v}$ be orthogonal unit vectors with the orientation of $\vec{u}$ being $\theta$. We define the *height* of a point $x$ with respect to $\theta$, written $h_\theta(x)$, as $\lambda$

such that $x = \lambda\vec{u} + \phi\vec{v}$. For a line segment $s$ orthogonal to $\theta$, $h_\theta(s)$ defined as $h_\theta(x)$ for any point $x \in s$. A path is called *$\theta$-monotone* if its height with respect to $\theta$ is either non-increasing or non-decreasing along the entire path. The following lemma, whose proof we omit, is often used as a definition of $\theta$-monotone (see e.g. [15]).

**Lemma 1** *A simple path is $\theta$-monotone if and only if it is $\{\,\theta + 90^\circ\,\}$-convex.*

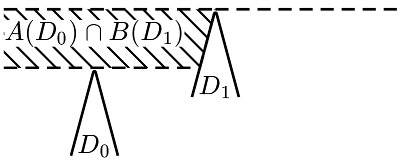
Lemma 1 has the restriction that it only applies to simple paths. Since every connected subset of an $\mathcal{O}$-convex path is $\mathcal{O}$-convex, we can convert a non-simple $\mathcal{O}$-convex subpath to a simple one by removing spikes. For any set of line orientations $\mathcal{O} \subset [0, 180)$, we use $\mathcal{O}^\perp \subset [0, 360)$ to denote the set of perpendicular orientations.

**Corollary 1** *Let $x$ and $y$ be two points in a simple polygon $P$. $x$ is $\mathcal{O}$-visible to $y$ if and only if there is a path from $x$ to $y$ inside $P$ that is $\theta$-monotone for each $\theta \in \mathcal{O}^\perp$.*

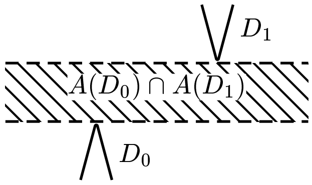
We next consider partitions of simple polygons induced by certain line segments. A maximal line segment not intersecting the exterior of the polygon is called a *chord*. An *oriented chord* is defined to be a pair $\delta = (\gamma, \theta)$ where $\gamma$ is a chord of $P$, and $\theta$ is one of the two orientations perpendicular to $\gamma$. With a slight abuse of notation we sometimes use $\delta$ to mean the unoriented chord $\gamma$. Each oriented chord of a simple polygon partitions the polygon into two (possibly non-simple) polygons. Given an oriented chord $\delta = (\gamma, \theta)$, let $A(\delta)$ (respectively $B(\delta)$) denote the region of $P$ intersected first by a ray from $\mathrm{int}(\gamma)$ with orientation $\theta$ (respectively $(\theta + 180^\circ)$). By convention $\delta$ is included in $A(\delta)$ but not in $B(\delta)$. If $x \in A(\delta)$ (respectively $B(\delta)$), we say that $x$ is *above* (respectively *below*) $\delta$. If a point $x$ is in $B(\delta)$, $B(\delta) \cup \delta$, $A(\delta)$, and $A(\delta) \setminus \delta$, respectively, we write $x \prec \delta$, $x \preceq \delta$, $x \succeq \delta$, and $x \succ \delta$, respectively. We use e.g. $\delta \prec x$ as alternative notation for $x \succ \delta$.

If $\delta = (\gamma, \theta)$ is an oriented chord, we use $\theta(\delta)$ (called the *orientation* of $\delta$) to refer to $\theta$. We call an oriented chord $\delta$ an *$\mathcal{O}$-chord* if $\theta(\delta) \in \mathcal{O}^\perp$. We are interested in particular $\mathcal{O}$-chords, namely those tangent to the boundary of the polygon. If $\tau$ is a local maximum (vertex or line segment) in the boundary of a polygon $P$ with respect to some $\theta$ in $\mathcal{O}^\perp$ and a ray from $\tau$ with orientation $\theta$ is inside $P$ in the neighbourhood of $\tau$ then $(\tau, \theta)$ is called a *dent*. Given a dent $D = (\tau, \theta)$, we call the oriented chord tangent to $\tau$ with orientation $\theta$ the *dent chord* of $D$, written $\vec{D}$. A given vertex may be part of more than one dent, but a given edge may be part of at most one. Given a dent $D = (\tau, \theta)$, $\theta(D)$ (the *orientation* of $D$) denotes $\theta$; we sometimes use the term dent and the notation $D$ to refer to $\tau$. We use $A(D)$ to denote $A(\vec{D})$, and $B(D)$ to denote $B(\vec{D})$.

In the degenerate case, two or more dent chords $\vec{D}_0$ and $\vec{D}_1$ may be coincident (i.e. have the same endpoints). We assume that the dents in the boundary of the polygon are ranked in some arbitrary but fixed way, and that $D_0$ is the lower ranked of the two. If the two dents have the same orientation, we assume that $\vec{D}_0$ stops at $D_1$ and consider there to be a zero width region between $\vec{D}_0$ and $\vec{D}_1$ that is above $D_0$ but below $D_1$ (see Figure 1(a)). If $\theta(D_0) = \theta(D_1) + 180^\circ$ then we consider there to be a zero width region between $\vec{D}_0$ and $\vec{D}_1$ that is above both $D_0$ and $D_1$ (see




(a) Same Orientation




(b) Opposite Orientations

Fig. 1. The two cases of zero width regions between coincident dent chords.

Figure 1(b)). We may further subdivide $B(D)$. The dent chord $\vec{D}$ can be thought of as two disjoint collinear line segments from $D$ to the polygon boundary. We define $B_l(D)$ (respectively $B_r(D)$) to be the half-polygon induced by $D$ containing the polygon edge clockwise (respectively counterclockwise) from $D$ (see Figure 2).

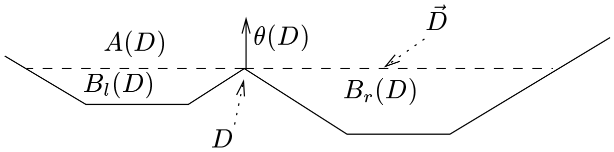



Fig. 2. A dent, and the three subpolygons induced by it.

We call $B_l(D)$ and $B_r(D)$ the two *sides* of $B(D)$. For any point $p \in B(D)$, we use $B_p(D)$ to denote the side of $B(D)$ containing $p$ and $B_{\bar{p}}(D)$ denote the other side of $B(D)$. A *separating dent* for two points $x$ and $y$ is a dent $D$ such that $x \in B_{\bar{y}}(D)$. In the rest of this paper we use $x$ *sees* $y$ to mean $x$ is $\mathcal{O}$-visible to $y$.

**Lemma 2** *$x$ sees $y$ if and only if there is no separating dent for $x$ and $y$.*

**Proof.** If there is a separating dent for $x$ and $y$ with orientation $\theta$ then no path from $x$ to $y$ can be $\theta$-monotone, hence by Lemma 1, $x$ cannot see $y$. Suppose that $x$ does not see $y$. Let $S$ be the shortest path from $x$ to $y$ inside the polygon. Since $S$ is not $\mathcal{O}$-convex, there must be some $\theta \in \mathcal{O}^{\perp}$ such that $S$ has a local maximum with respect to $\theta$. It is well known (see e.g. [12]) that shortest paths inside polygons are polygonal chains with vertices other than endpoints being vertices of the polygon, thus this local maximum must be a dent. Since $S$ is the shortest path from $x$ to $y$, $x$ and $y$ must be on opposite sides of this dent. □

**Corollary 2** *If a point $x$ sees points $y$ and $z$, $x$ must be above any separating dent for $y$ and $z$.*

We say that two points $p$ and $q$ in a polygon $P$ are *dent equivalent* if for any dent $D$ in $P$, for any $R \in \{ A(D), B_l(D), B_r(D) \}$, $p \in R \Leftrightarrow q \in R$. We call the maximal equivalence classes of this relation *dent cells*.

**Corollary 3** *If two points in a polygon are in the same dent cell then they see the same set of points.*

A *quotient graph* $H$ of a graph $G$ is a graph whose vertices are equivalence classes of vertices of $G$ and whose edges are pairs of equivalence classes with adjacent members. We have seen that all of the points in a given dent cell see the same subset of the polygon (the converse holds only for finite sets of orientations $\mathcal{O}$). We use these equivalence classes to define a labeled quotient graph of the PVG. Given two dent cells $K_0$ and $K_1$, we say that $K_0$ *sees* $K_1$ if the points in $K_0$ see the points in $K_1$. We define the *cell visibility graph* (CVG) of a polygon as follows: the vertices of a CVG are the cells of the dent decomposition, labeled with cardinalities of those cells (i.e. whether the cells contain a single point or an infinite number), and there is an edge between two vertices if and only if the corresponding dent cells see one another. If the cardinality of $\mathcal{O}$ is finite, then the CVG is a finite combinatorial representation of the point visibility graph. Simple arguments using brute force algorithms on the CVG show that if $\mathcal{O}$ is finite, $\mathcal{O}$-convex cover and $\mathcal{O}$-star cover (finding the minimum cardinality set of points that see the entire polygon between them) are both solvable in polynomial time if the number of covering polygons is fixed (for details, see [3]).

## 3. Weakly Triangulated PVGs

Many problems that are NP-Hard on general graphs become tractable on suitable classes of graphs. Perhaps the most well-known class of graphs used in obtaining polynomial upper bounds is the class of perfect graphs. Since there is ambiguity inherent in speaking of the set of dent orientations of a polygon, for precision we define a *visibility instance* as a polygon $P$ and a set of orientations $\mathcal{O}$. In this section we show that if the set of dent orientations of a visibility instance is restricted sufficiently, the resulting PVG is weakly triangulated, hence any finite induced subgraph is perfect. For a restricted class of visibility instances, this will provide a duality between $\mathcal{O}$-hidden set (finding the largest cardinality set of points with no pair $\mathcal{O}$-visible) and $\mathcal{O}$-convex cover, and a polynomial algorithm for both problems. Culberson and Reckhow[4] have shown that CVGs of orthogonal polygons with up to 2 dent orientations in $\{0°, 90°, 180°, 270°\}$ are comparability graphs and Motwani, Raghunathan, and Saran[14] extended this work to show that CVGs of orthogonal polygons with up to 3 dent orientations in $\{0°, 90°, 180°, 270°\}$ are weakly triangulated. Following the terminology of [4,14], let a *class $j$ visibility instance* be one with at most $j$ dent orientations. Our set of class 3 visibility instances includes the class 3 polygons of the previous authors as a special case. With a slight abuse of terminology, we use *cyclic class 3* set of orientations to mean a cyclic set of 3 orientations, or an arbitrary set of less than 3 orientations. We show that not all class 3 visibility instances have weakly triangulated PVGs, but that cyclic class 3 visibility instances do.

Our arguments will have the general flavour of showing that the presence of $k$-holes or $k$-antiholes in the PVG requires certain configurations of dents. We start by considering some consequences of the relationship between $\mathcal{O}$-visibility and monotonicity. The following lemma is useful in characterizing when two dents cannot have the same dent orientation. We write $x \sim y$ if $x$ is $\mathcal{O}$-visible to $y$, $x \not\sim y$ otherwise.

**Lemma 3** *Let $x$ be a point and $\delta$ an $\mathcal{O}$-chord with orientation $\theta$ such that for some $y \in \delta$, $x \sim y$.*

*(a) $x \prec \delta$ iff $h_\theta(x) < h_\theta(y)$, and*

*(b) $x \succ \delta$ iff $h_\theta(x) > h_\theta(y)$.*

**Proof.** We consider only case (a); the other is symmetric. Suppose $x \prec \delta$. Let $S$ be a simple $\mathcal{O}$-convex path from $y$ to $x$. We know from Lemma 1 that $S$ must either be non-decreasing or non-increasing with respect to $\theta$. If it is non-decreasing then it can never cross $\delta$, hence can never reach $x$. Since $x \notin \delta$, it follows $h_\theta(x) < h_\theta(y)$. Conversely, suppose $h_\theta(x) < h_\theta(y)$. Again consider a simple $\mathcal{O}$-convex path $S$ from $y$ to $x$. Since $S$ must be non-increasing, it follows that $x \preceq \delta$. But if $x \in \delta$ then $h_\theta(x) = h_\theta(y)$, so $x \prec \delta$. □

If two dents or oriented chords $D$ and $D'$ have the same orientation, we write $D \parallel D'$; conversely if they have different orientations we write $D \nparallel D'$ (note that this is stronger than the usual usage of the symbol "$\parallel$"). By noting that if an $\mathcal{O}$-convex path intersects an oriented chord, both endpoints of the path see the point of intersection, we have the following corollaries that will be useful below.

**Corollary 4** *Let $\delta_1$ and $\delta_2$ be $\mathcal{O}$-chords. If $x \sim y$, $x \prec \delta_1 \preceq y$, and $y \prec \delta_2 \preceq x$, then $\delta_1 \nparallel \delta_2$.*

**Corollary 5** *Let $S$ be an $\mathcal{O}$-convex path from $x$ to $y$. Let $\delta$ be an $\mathcal{O}$-chord that intersects* $\operatorname{int} S$. *If $x \prec \delta$ then $\delta \preceq y$.*

Suppose that $D$ is a dent, and $a$, $b$, and $c$ are sets of points such that $a \subseteq B(D)$, $b \subseteq B_{\bar{a}}(D)$, and $c \subseteq A(D)$. We denote this situation by

$$D = \frac{c}{a \mid b}.$$

Let $\mathcal{D}$ be a collection of not necessarily distinct dents. An *incompatibility graph* $G$ of $\mathcal{D}$ is a graph where the vertices of $G$ are the elements of $\mathcal{D}$, and $\{D_i, D_j\}$ is an edge of $G$ only if $D_i$ is constrained not to be the same orientation as $D_j$. The chromatic number of an incompatibility graph is a lower bound on the the minimum number of distinct dent orientations present in $\mathcal{D}$. In the following, we take all vertex indices modulo $k$. Furthermore, where $a < b$ we take the ordered set $\langle v_b \dots v_a \rangle$ to mean $\langle v_b, v_{b+1} \dots v_0, v_1 \dots v_a \rangle$. Let $\| i - j \|$ denote the "modular difference" between $i$ and $j$, that is the smallest nonnegative $j'$ such that $(i + j' \equiv j \pmod{k}) \vee (j + j' \equiv i \pmod{k})$.

**Theorem 1** *A visibility instance whose PVG contains a $k$-antihole, $k > 4$, must have at least $\lceil k/2 \rceil$ dent orientations.*

**Proof.** Let the vertices of a $k$-antihole $C$ be labeled cyclically $v_0 \ldots v_{k-1}$. We know that there must be a separating dent $D_i$ for each pair $(v_i, v_{i+1})$. We will prove the theorem by establishing the following implication.

$$\| i - j \| > 1 \implies D_i \not\parallel D_j \,. \tag{1}$$

From (1) it will follow that the incompatibility graph of the set of dents separating adjacent elements of a $k$-antihole contains a (not necessarily induced) $k$-anticycle; i.e. there exist $k$ nodes in the complement graph with at most a cycle connecting them. A $k$-cycle requires at least $\lceil k/2 \rceil$ cliques to cover it; hence the chromatic number of a graph containing a $k$-anticycle is no less than $\lceil k/2 \rceil$.

We now argue that (1) holds. Let $\beta_i$ denote the set of vertices that see both $v_i$ and $v_{i+1}$. Two vertices $v_i$ and $v_j$ of the $k$-antihole are $\mathcal{O}$-visible if and only if $\| i - j \| \neq 1$, hence $\beta_i = \langle\, v_{i+3} \ldots v_{i-2} \,\rangle$. By Corollary 2 every vertex in $\beta_i$ must be above $D_i$; i.e.:

$$D_i = \frac{\beta_i}{v_i \;\big|\; v_{i+1}} \,. \tag{2}$$

As a consequence of (2), we have

$$v_j \prec D_i \implies i - 1 \leq j \leq i + 2 \,. \tag{3}$$

We now consider the pairwise compatibility of the separating dents $\{\, D_i \mid 0 \leq i < k \,\}$. Let $i$ and $j$ be vertex indices such that $\| i - j \| > 1$. There are two cases.

**Case 1.** Suppose $v_j \prec D_i$. From (3), $j = i + 2$. It follows that $\beta_j = \langle\, v_{i+5} \ldots v_i \,\rangle$. Since there are at least 5 vertices in the antihole, $v_i \in \beta_j$, hence $v_i \succeq D_j$. Furthermore, $j + 1 = i + 3$, so $v_{j+1} \in \beta_i$, and hence $v_{j+1} \succeq D_i$. We know from (2) that $v_{j+1} \prec D_j$ and $v_i \prec D_i$, so we have the following relationships:

$$\begin{aligned} v_{j+1} \prec D_j &\preceq v_i \\ v_i \prec D_i &\preceq v_{j+1}. \end{aligned}$$

It follows by Corollary 4 that $D_i \not\parallel D_j$.

**Case 2.** Otherwise $v_j \succeq D_i$. By (2),

$$v_{i+1} \prec D_i \preceq v_j. \tag{4}$$

If $v_{i+1} \prec D_j$ then by (3), $i = j + 2$. By the preceding argument, $D_i \not\parallel D_j$. Otherwise we have:

$$v_j \prec D_j \preceq v_{i+1}. \tag{5}$$

It follows by (4), (5) and Corollary 4 that $D_i \not\parallel D_j$. $\square$

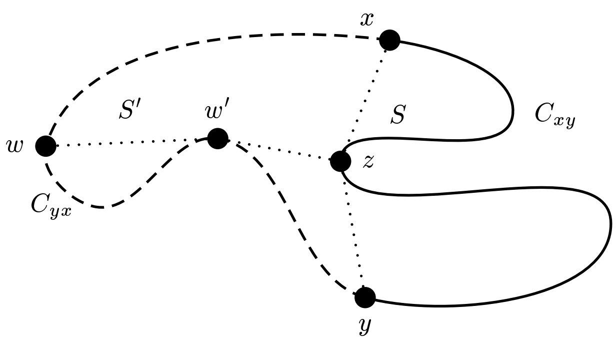



Fig. 3. Illustration of proof of the crossing lemma.

We can strengthen the above theorem in the special case of a 6-antihole, but in order to do so we will need the following generalization of a lemma contained in [14]. An embedding of a graph $G$ in a polygon $P$ such that each edge is drawn as a polygonal $\mathcal{O}$-convex path inside $P$ is called an *instantiation* of $G$. An instantiation is called *planar* if the instantiated edges do not cross (i.e. intersect other than at their endpoints). In the following discussion we use the names of vertices and edges of the graph for the corresponding elements of the instantiation.

**Lemma 4 (The Crossing Lemma)** *Let $C$ be a hole, $|C| \geq 4$. In any instantiation of $C$, there exists a pair of non-adjacent edges that cross.*

**Proof.** We first note that an instantiation where only adjacent edges cross implies a planar instantiation, since each crossing can be replaced with a vertex of a (new) chordless cycle of the same size. From the chordlessness of $C$ we know no vertex of $C$ lies on the instantiation of a non-adjacent edge. We can thus assume without loss of generality that a planar instantiation of a cycle forms a simple polygon.

Suppose there were a planar instantiation of $C$. Let $Q$ be the simple polygon bounded by a planar instantiation of $C$. Since the boundary of $Q$ between vertices of $C$ consists of $\mathcal{O}$-convex paths, dents in the boundary of $Q$ can only occur at vertices of $C$.

Consider two non-adjacent vertices of $C$, $x$ and $y$. Let the non-empty chain of $C$ clockwise from $x$ to $y$ be $C_{xy}$. Let the non-empty chain of $C$ clockwise from $y$ to $x$ be $C_{yx}$. Since $C$ is chordless, $x \not\sim y$. Let $S$ be the geodesic (shortest path) in $Q$ between $x$ and $y$. Recall from the proof of Lemma 2 that a geodesic between two non-visible points must be supported by a dent in the polygon boundary. Let $z$ be some vertex or edge of $C$ that is a supporting dent for $S$. Without loss of generality, assume $z \in C_{xy}$. Let $w$ be some vertex in $C_{yx}$. By the chordlessness of $C$, $w$ does not see $z$. Let $S'$ be a geodesic between $w$ and $z$, and let $w'$ be the closest (along $S'$) supporting dent of $S'$ to $z$ (see Figure 3). All of $C_{xy}$ is on the same side of $S$ as $z$, hence $w'$ must be the instantiation of some vertex in $C_{yx}$. There is no dent on the subpath of $S'$ between $w'$ and $z$, hence $w'$ must be $\mathcal{O}$-visible from $z$, but this is a contradiction of the chordlessness of $C$. It follows that there cannot be a planar instantiation of $C$. □

**Lemma 5** *Let $D_0$ be a separating dent for $x$ and $y$. Let $D_1$ be a separating dent for $x$ and $z$. If $D_0 \parallel D_1$ and $y \sim z$ then $D_0 = D_1$.*

**Proof.** If $D_0 \neq D_1$ then either $B(D_0) \subseteq B_{\bar{x}}(D_1)$ or $B(D_1) \subseteq B_{\bar{x}}(D_0)$. In either case, this is contradiction since $x \in B(D_0) \cap B(D_1)$. □

It is often useful for the sake of concreteness to assume that dent orientations take on particular values. Since for any $\phi$ there is an affine transformation that maps horizontal lines to horizontal lines and (non-horizontal) lines with orientation $\phi$ to vertical lines (see e.g. [6]), Rawlins[17] observed that one may assume that any particular set of line orientations contains a pair of orientations that are perpendicular. By considering a further appropriate reflections, we have the following:

**Observation 2** *For any visibility instance $I = (P, \mathcal{O})$, for any two dents $D_1$ and $D_2$ of $I$ with non-parallel dent chords, there exists an affine transform $\Xi$ such that*

*(a) $I' = (\Xi(P), \Xi(\mathcal{O}))$ has the same PVG as $I$,*

*(b) $\theta(\Xi(D_1)) = \theta(\Xi(D_2)) + 90°$*

In order to apply Observation 2, we require dents with distinct, non-opposite orientations. The following observation is useful in this context.

**Observation 3** *Let $D_1$ and $D_2$ be dents. If $B(D_1) \cap B(D_2) \neq \emptyset$ and $A(D_1) \cap A(D_2) \neq \emptyset$, then $\theta(D_1) \neq \theta(D_2) + 180°$.*

The following technical lemma will prove useful below.

**Lemma 6** *Let $\delta_0$, $\delta_1$ and $\delta_2$ be oriented chords with distinct orientations through a point $q$. Let $T = \{ x, y, z \}$ be a set of points such that*

*(a) $T \subset B(\delta_0)$,*

*(b) Each $w \in T$ is in a unique cell $K_w$ of the the arrangement of the chords $\delta_i$, and*

*(c) The cyclic order of the cells $K_j$ about $q$ is $\langle K_x, K_y, K_z \rangle$.*

*For any dent $D$ with same orientation as $\delta_0$, if $z \in B_x(D)$ then $y \notin B_{\bar{x}}(D)$.*

**Proof.** Let $D$ be some dent with the same orientation as $\delta_0$. If $q \prec D$ all of the points must be below and on the same side of $D$. Otherwise, $q \succeq D$. Suppose $z \in B_x(D)$. Let $\gamma_x$ be the segment of $\vec{D}$ bounding $B_x(D)$. Since both $\delta_1$ and $\delta_2$ split $x$ and $z$, they must both intersect $\gamma_x$. It follows that $y$ cannot be in $B_{\bar{x}}(D)$. □

**Lemma 7** *A polygon with only three dent directions cannot have a 6-antihole in its PVG.*

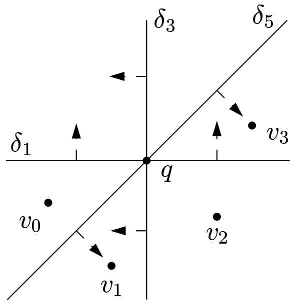



Fig. 4. Illustrating the proof of Lemma 7

**Proof.** Consider a 6-antihole in the PVG of a class 3 visibility instance. As in the proof of Theorem 1 let the separating dent for vertices $v_i$ and $v_{i+1}$ be $D_i$. From the proof of Theorem 1, we know that the dent incompatibility graph itself contains a six anticycle. Since we assume that the six antihole is generated by a visibility instance with only three dent orientations, it follows that the dent incompatibility graph must be 3-colourable. The only way to 3-colour this graph is to assign the same colour to successive pairs of vertices along the anticycle. Without loss of generality, assume that $D_0$ is assigned the same colour (i.e. is the same orientation as) $D_1$, $D_2$ is assigned the same colour as $D_3$, and so on. It follows that the separating dents for the the 6-antihole are divided into equivalence classes as follows:

| Orientation | Dents | |
|---|---|---|
| $\theta_1$ | $\dfrac{v_3, v_4}{v_0 \mid v_1}$ | $\dfrac{v_4, v_5}{v_1 \mid v_2}$ |
| $\theta_2$ | $\dfrac{v_5, v_0}{v_2 \mid v_3}$ | $\dfrac{v_0, v_1}{v_3 \mid v_4}$ |
| $\theta_3$ | $\dfrac{v_1, v_2}{v_4 \mid v_5}$ | $\dfrac{v_2, v_3}{v_5 \mid v_0}$ |

A 6-antihole contains several 4-holes. Consider the chordless cycle $C = \langle v_0, v_4, v_1, v_3 \rangle$. By the Crossing Lemma, there must some point $q$ where two non-adjacent edges of $C$ cross. The point $q$ sees each of $v_0, v_1, v_3$, and $v_4$. By Lemma 5 and Corollary 2 we can deduce the existence of the following three dents:

$$\Delta_1 = \frac{v_3, v_4, v_5, q}{v_1 \mid v_0, v_2}, \quad \Delta_3 = \frac{v_5, v_0, v_1, q}{v_3 \mid v_2, v_4}, \quad \Delta_5 = \frac{v_1, v_2, v_3}{v_5 \mid v_4, v_0} \tag{6}$$

The point $q$ must be the intersection of edges $v_0v_3$ and $v_1v_4$, since otherwise $q$ would have to be both above and below $\Delta_5$. Let $\delta_i$ denote the oriented chord through $q$ with the same orientation as $\Delta_i$. By Lemma 3, we can deduce the following relationships between the vertices of $C$ and the chords $\delta_i$ (where $A$ (resp. $B$) indicates

$v_j \in A(\delta_i)$ (resp. $v_j \in B(\delta_i)$).

| | $v_0$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
|---|---|---|---|---|---|
| $\delta_1$ | $B$ | $B$ | $B$ | $A$ | $A$ |
| $\delta_3$ | $A$ | $A$ | $B$ | $B$ | $B$ |
| $\delta_5$ | $B$ | $A$ | | $A$ | $B$ |

By Observation 2, we may assume that $\theta(\delta_1) = 90°$ and $\theta(\delta_3) = 180°$. In order that $v_0$ and $v_4$ both be below $\delta_5$, it must be the case that $270 < \theta(\delta_5) < 360$ (see Figure 4). By Lemma 6, $\Delta_1$ cannot separate $v_1$ from $v_0$ and $v_2$, a contradiction. $\square$

**Lemma 8** *If the PVG of a visibility instance $(P, \mathcal{O})$ contains a $k$-hole, $k \geq 5$, then*

*(a) $(P, \mathcal{O})$ has at least 3 dent orientations, and*

*(b) If it has exactly three dent orientations, then the three dent orientations are acyclic.*

**Proof.** Let $k$ be the smallest integer greater than 4 such that there is $k$-hole in the $\mathcal{O}$-PVG of $P$. Let $C$ be a $k$-hole in the $\mathcal{O}$-PVG of $P$. By the Crossing Lemma there are two non-adjacent edges that cross in any instantiation of $C$. Let $v_0v_3$ and $v_1v_2$ be two edges whose instantiations cross; let $q$ be a point where the two edges cross; $q$ is visible from $v_0$, $v_1$, $v_2$, and $v_3$. Without loss of generality let the cyclic order of $v_0$, $v_1$, $v_2$, $v_3$ around $C$, possibly with intervening vertices, be $\langle v_0, v_1, v_2, v_3 \rangle$. There must be at least one more vertex in $C$, without loss of generality between $v_2$ and $v_3$ (see Figure 5).

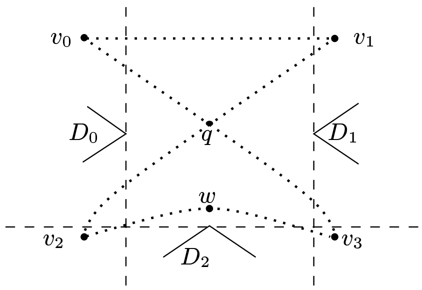



Fig. 5. Vertex layout of a $k$-cycle, $k \geq 5$ in a class 3 visibility instance.

By Corollary 2 and Corollary 5, and the chordlessness of $C$, the following dents must exist:

$$D_0 = \frac{v_1, v_3, q}{v_0 \mid v_2} \qquad D_1 = \frac{v_0, v_2, q}{v_1 \mid v_3} \qquad D_2 = \frac{v_0, v_1, q}{v_2 \mid v_3}$$

By Corollary 4, the orientations of these three dents are pairwise incompatible, so the polygon must contain dents of at least three distinct orientations.

In order to obtain a contradiction, suppose that the three dent orientations in $P$ are cyclic. By Observations 2 and 3 we may assume that the orientation of $D_2$

is $90°$ and the orientation of $D_0$ is $0°$. Since the dent orientations are cyclic, the orientation of $D_1$ must be $180° + \phi$ where $0° \leq \phi \leq 90°$. Since the case of $\phi = 90$ is equivalent by relabeling to the case of $\phi = 0$, we may assume that $0° \leq \phi < 90°$.

Let the chain of vertices from $v_2$ to $v_3$ be labeled $\langle w_1 \ldots w_{k-4} \rangle$. Let $w'$ denote the vertex after $w_1$ in a traversal of $C$ in the cycle order that reaches the vertices $\{ v_0, v_1, v_2, v_3 \}$ in the order $\langle v_0, v_1, v_2, v_3 \rangle$. We first establish the following:

$$w_1 \sim q \ \vee \ w' \sim q. \tag{7}$$

To see that (7) must hold, we consider two possible cases.

1. Suppose $k = 5$; then $w' = v_3$ and $v_3$ sees $q$.
2. Suppose $k > 5$. Let $w_j$ be the first of the tuple $\langle w_1, \ldots w_{k-4}, v_3 \rangle$ that sees $q$. If neither $w_1$ nor $w_2$ sees $q$ then $C' = \langle q, v_2, w_1, w_2, \ldots w_j \rangle$ forms a hole of size $k'$, $5 \leq k' < k$, generated by the same set of dent orientations, and this contradicts our assumption that $C$ was the minimal counterexample.

We now consider the two possible cases of (7):

**Case 7a:** $w_1 \sim q$. From the chordlessness of $C$ and Corollary 5 we know that the following dents must exist:

$$D_4 = \frac{q, v_3}{w_1 \mid v_0}$$

$$D_5 = \frac{q, v_2}{w_1 \mid v_1}.$$

By Corollary 4, $D_4 \parallel D_0$ and $D_5 \parallel D_1$. Let $\delta_i$ be an oriented chord through $q$ with the same orientation as dent $D_i$ (see Figure 6).

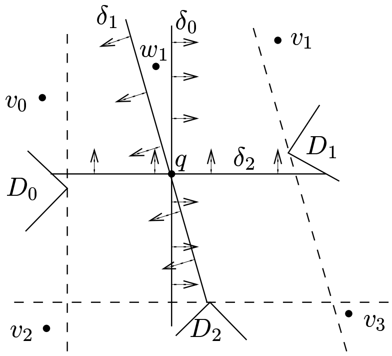



Fig. 6. The oriented chord $\delta_i$ has the same orientation as $D_i$.

By Lemma 3 $w_1$ must be below $\delta_0$ and $\delta_1$. Since the orientation of $\delta_1$ makes an angle of at least $90°$ with the other two orientations, $w_1 \succ \delta_2$. By Lemma 5,

$$D_0 = D_4 = \frac{q, v_1, v_3}{w_1, v_2 \mid v_0}.$$

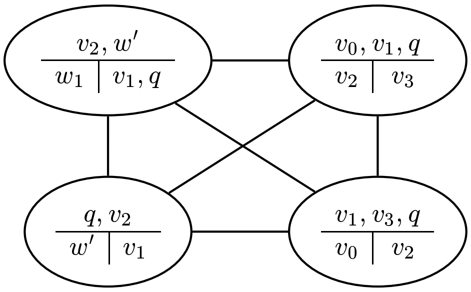



Fig. 7. Incompatibility graph of dents induced by a $k$-hole if $w_1 \not\sim q$.

By Lemma 6, $D_0$ cannot separate $v_0$ from $w_1$ and $v_2$, a contradiction.

**Case 7b:** $w_1 \not\sim q$ & $w' \sim q$. Consider a separating dent $D_q$ for $w_1$ and $q$. The points $v_2$ and $w'$ see both $w_1$ and $q$, so by Corollary 2 must be above $D_q$. By Lemma 3 and Lemma 2, $v_1$ must be below and on the same side of $D_q$ as $q$.

$$D_q = \frac{v_2, w'}{w_1 \mid v_1, q}$$

From the chordlessness of $C$, there must be some separating dent $D_w$ for $w'$ and $v_1$. From Corollary 2 the point $q$ must be above $D_w$; it follows from Corollary 5 that $v_2$ must also be above $D_w$, i.e.

$$D_w = \frac{q, v_2}{w' \mid v_1}$$

By Corollary 4, the dents $D_0$, $D_2$, $D_w$ and $D_q$ are pairwise incompatible (see Figure 7). This implies there are more than 3 dent orientations in $P$, a contradiction. □

We now give a construction that shows that the previous lemma is as strong as possible.

**Lemma 9** *For any acyclic class 3 set of orientations, there exists a polygon with dent orientations only in that set and a 5-hole in its PVG.*

**Proof.** Suppose we have a set of three dent orientations $\{\theta_0, \theta_1, \theta_2\}$ contained in an open half-circle. We argue that the construction of Figure 8 can be modified so that it has exactly this set of dent orientations (for appropriate choice of $\mathcal{O}$). Without loss of generality suppose that $\theta_0 < \theta_1 < \theta_2$. Without loss of generality, we may assume that $\theta_0 = 0° + \phi$ and $\theta_2 = 360 - \phi$ for some $0 < \phi < 90$. By appropriate horizontal scaling, we can ensure that the PVG of Figure 8 is preserved with these dent orientations. Furthermore, since $\theta_1$ is between $\theta_0$ and $\theta_2$ we can tilt the bottom reflex edge so that the the third dent orientation of Figure 8 is $\theta_1$, also without changing the PVG. □

**Corollary 6** *Let $\mathcal{O}'$ be a set of orientations. If $|\mathcal{O}'| \geq 4$, then there exists a visibility instance with dent orientations only in $\mathcal{O}'$ and a 5-hole in its PVG.*

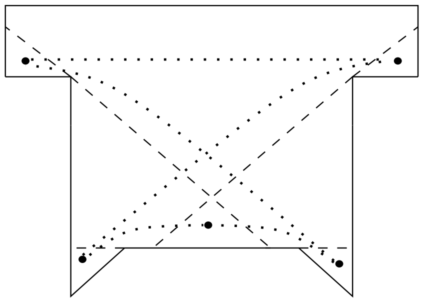


Fig. 8. A polygon with 3 dent orientations ($\mathcal{O} = \{0°, 45°, 90°, 135°\}$) and a chordless 5-cycle in its PVG.

**Proof.** Let $\mathcal{O}'$ be a set of orientations, $|\mathcal{O}'| \geq 4$.

1. Suppose $|\mathcal{O}'| = 4$. If the orientations in $\mathcal{O}'$ form two antipodal pairs, then we can apply Observation 2 and the construction for class 4 orthogonal polygons from [14]. Otherwise, consider a line through the origin, along some orientation $\phi \in \mathcal{O}'$ such that $\phi + 180° \notin \mathcal{O}'$. Two of the orientations in $\mathcal{O}'$ must be on one side of this line, and one of the orientations on the other. The two orientations on one side of the line, along with $\phi$, form an acyclic class 3 set of orientations, and we can apply the construction of Lemma 9.

2. Otherwise $|\mathcal{O}'| \geq 5$. Consider a line through the origin that is not along any orientation in $\mathcal{O}'$. By the pigeonhole principle, one of the halfplanes induced by this line must contain at least 3 orientations so we can again apply the construction of Lemma 9. □

We have now found enough forbidden subgraphs for cyclic class 3 visibility instances to establish the main result of the paper:

**Theorem 4** *If $(P, \mathcal{O})$ is a cyclic class 3 visibility instance then the $\mathcal{O}$-PVG of $P$ is weakly triangulated.*

**Proof.** Let $G$ be the PVG of a cyclic class 3 visibility instance. From Lemma 8, $G$ has no 5-antihole (since a 5-hole is a 5-antihole), or $k$-hole for $k \geq 5$. From Lemma 7, $G$ has no 6-antihole. From Theorem 1, $G$ has no $k$-antihole, $k > 6$. □

Theorem 4 provides both an interesting structural duality about cyclic class 3 visibility instances and an algorithmic result. Motwani et al.[14] noted that since the CVGs of class 3 orthogonal polygons are perfect, the maximum hidden set in a class 3 orthogonal polygon is the same size as the minimum convex cover. This result generalizes to the following:

**Corollary 7** *The maximum $\mathcal{O}$-hidden set of a cyclic class 3 visibility instance is the same size as the minimum $\mathcal{O}$-convex cover.*

**Proof.** Let $(P, \mathcal{O})$ be a cyclic class 3 visibility instance. By Theorem 4, the $\mathcal{O}$-PVG $G_p$ of $P$ is weakly triangulated. Let $G_c$ be the $\mathcal{O}$-CVG of $P$. Since $G_c$ is an induced subgraph of $G_p$, $G_c$ is also weakly triangulated. Recall that $\alpha(G)$ is maximum independent set in $G$ and $k(G)$ is the minimum number of cliques to cover $G$. Finite weakly triangulated graphs are perfect, so $\alpha(G_c) = k(G_c)$. From the fact that $G_c$ is a quotient graph of $G_p$ and the fact that PVGs are reflexive (i.e. each equivalence class is a clique), it follows that $\alpha(G_p) = \alpha(G_c) = k(G_c) = k(G_p)$ □

Note that the several steps here are necessary since because of their uncountable nature, we do not know that weakly triangulated PVGs are perfect. On the other hand, since a 5-cycle is not perfect, the examples of Lemma 9 and Corollary 6 have PVGs that are not only not weakly triangulated, but also not perfect.

Raghunathan[16] has given an algorithm that finds a maximum clique and a minimum colouring of a weakly triangulated graph $G = (V, E)$ in $O(ev^2)$ time where $e = |E|$ and $v = |V|$. Treating $|\mathcal{O}|$ as a constant, we have the following.

**Corollary 8** *Let $(P, \mathcal{O})$ be a cyclic class 3 visibility instance with $P$ having $n$ vertices. $\mathcal{O}$-convex cover and $\mathcal{O}$-hidden set can be computed on $P$ in $O(n^8)$ time.*

**Proof.** If a polygon $P$ has $n$ edges, then the CVG of $P$ has $O(n^2)$ vertices and $O(n^4)$ edges. The $\mathcal{O}$-CVG of $P$ can be computed in $O(n^4)$ time in a fairly straightforward manner (see e.g. [3]). The complement of a weakly triangulated graph is also weakly triangulated, so we can use Raghunathan's algorithm on the graph theoretic complement of the CVG to find a maximum independent set and a minimum clique cover of the CVG. It follows that in $O(n^8)$ time we can find a maximum independent set and minimum clique cover in the CVG. We can map an independent set in the CVG back into a hidden set in $O(n)$ time by simply choosing one point from each cell. We can merge the sets of cells corresponding to a clique in the CVG into an $\mathcal{O}$-convex polygon in time proportional to the number of cells by a depth first walk in the dent decomposition. □

## 4. Conclusions

This paper follows [4,14] in using the notion of dent orientation to characterize which visibility instances have weakly triangulated PVGs. We show, however, that for more general kinds of visibility, the number of dent orientations is not sufficient for this characterization. We show that to guarantee a weakly triangulated PVG, not only must a polygon have a maximum of 3 dent orientations, but if it does have 3 dent orientations, these three orientations must be cyclic.

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