# Lecture 13: Algebraic and Union Types

David Bremner

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# define-type

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btl
```
(define-type BT
  [mt]
  [node (v : Number) (l : BT) (r : BT)])
```

- Binds the name `BT` in the current type environment
- Supports recursion
- Defines two variants `mt` and `node`
- What are the types of `mt` and `node`?

# Generated bindings

## Constructors

```
(mt : ( -> BT))
(node : (Number BT BT -> BT))
```

## Predicates

```
(mt? : (BT -> Boolean))
(node? : (BT -> Boolean))
```

## Accessors

```
(node-v : (BT -> Number))
(node-l : (BT -> BT))
(node-r : (BT -> BT))
```

# The problematic ones

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```
(node-v : (BT -> Number))
(node-l : (BT -> BT))
(node-r : (BT -> BT))
```

bt2

```
(define (size-wrong (t : BT))
  (+ 1 (+ (size-wrong (node-l t))
          (size-wrong (node-r t)))))
(size-wrong (mt))
```

# Safely accessing variants

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bt3

```
(define (size-pm t)
  (type-case BT t
    [(mt) 0]
    [(node v l r) (+ 1 (+ (size-pm l) (size-pm r)))]))
(size-pm (mt))
```

# Typechecking `type-case`

Each `define-type` *extends* the type checker.

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$$\frac{\Gamma \vdash \texttt{e} : BT \quad \Gamma \vdash \texttt{e1} : T \quad \Gamma[V \leftarrow \mathrm{Num}, L \leftarrow \mathrm{BT}, R \leftarrow \mathrm{BT}] \vdash \texttt{e2} : T}{\Gamma \vdash \texttt{(type-case BT e [(mt) e1][(node V L R) e2])} : T}$$

- can be automatically generated

# Space usage for algebraic data type

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- Can desugar pattern matching into cond
- Need some level of “local” tags for predicates

bt4

```
(define (size-pm-ds (t : BT))
  (cond
    [(mt? t) 0]
    [(node? t)
     (let ([v (node-v t)]
           [l (node-l t)]
           [r (node-r t)])
       (+ 1 (+ (size-pm-ds l) (size-pm-ds r))))]))
```

# Retrofitted type systems

- A common (recent) strategy is to add a static type system to existing dynamically typed languages
- Examples include `Typescript` and `mypy` for python.
- The general idea is convert some run-time errors into compile-time errors.

# Example of retrofitted types: `typed/racket`

Type system accepts more, therefore sometimes needs more help.

trgood

```
(define f
  (lambda ([x : Number] [y : Number])
    (if x
        (+ y 1) (+ y 2))))
```

trbad

```
(lambda (x)
  (if x
      (+ x 1) (+ x 2)))
```

- still detects type-errors, but there are some subtle differences

# typed/racket accepts more code

Without annotation, inferred types are weaker.

trnew

```
(define f
  (lambda (x y)
    (cond
      [(string? y) (string-append "hello " y)]
      [(number? y) (if x  (+ y 1) (+ y 2))]
      [else (error 'f "unknown type")])))
```

# Another approach to variants

Variant information is lost to the type system.

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```
(node : (Number BT BT -> BT))
```

## Union types

In "typed/racket" we have another option.

bt5

```
(define-type-alias BT (U mt node))
(struct mt ())
(struct node ([v : Number] [l : BT] [r : BT]))
```

- what constructors, predicates, and accessors are defined?

# Using our union type

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bt6

```
(define t1
  (node 5
        (node 3
              (node 1 (mt) (mt))
              (mt))
        (node 7
              (mt)
              (node 9 (mt) (mt)))))

(node-l t1)  ; OK
;;(node-l (node-l t1)) ; not OK
```

# Computing the size

- Does the following typecheck? Why or why not?

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bt7

```
(define (size-tr [t : BT]) : Number
  (cond
    [(mt? t) 0]
    [(node? t) (+ 1
                  (size-tr (node-l t))
                  (size-tr (node-r t)))]))
```

# Predicates are special in typed/racket

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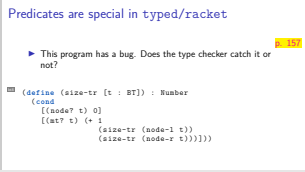
- This program has a bug. Does the type checker catch it or not?

bt8

```
(define (size-tr [t : BT]) : Number
  (cond
    [(node? t) 0]
    [(mt? t) (+ 1
                (size-tr (node-l t))
                (size-tr (node-r t)))]))
```

1. The type errors from typed/racket are much better that those from plait, at least here



# If-Splitting

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```
(define (size-tr [t : BT]) : Number
  (cond
    [(mt? t) …]
    [(node? t) …]))
```

$$\frac{\Gamma \vdash \mathtt{e} : (\mathsf{U}\ S_1 \ldots S_k) \quad \Gamma[e \leftarrow S_1] \vdash \mathtt{b}_1 : T_1 \ \ldots \ \Gamma[e \leftarrow S_k] \vdash \mathtt{b}_k : T_K}{\Gamma \vdash (\mathtt{cond}\ \ [(\mathtt{S}_1? \ e)b_1]...[(S_k? \ e)b_k]) : (\mathsf{U}\ T_1 \ldots T_k)}$$

# Option types and Unions 1/2

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div1

```
(define (safe-div num den)
  (if (zero? den)
      (none)
      (some (/ num den))))

(test (safe-div 42 0) (none))
(test (safe-div 42 12) (some 7/2))
```

# Option types and Unions 2/2

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div2

```
(define (safe-div2 [num : Number] [den : Number])
  (if (zero? den)
      #f
      (/ num den)))

(test (safe-div2 42 0) #f)
(test (safe-div2 42 12) 7/2)
```

- What is the type of `safe-div2`
- Arguably cleaner code than the `Option` case.
- Notice more annotation needed

# Polymorphic return values 1/2

Recall our (`plait`-style) rule for `if`:

$$\frac{\Gamma \vdash \texttt{C} : \text{Bool} \quad \Gamma \vdash \texttt{T} : W \quad \Gamma \vdash \texttt{E} : W}{\Gamma \vdash \texttt{(if C T E)} : W}$$

With union types we can generalize to

$$\frac{\Gamma \vdash \texttt{C} : \text{Bool} \quad \Gamma \vdash \texttt{T} : W \quad \Gamma \vdash \texttt{E} : Z}{\Gamma \vdash \texttt{(if C T E)} : (\mathsf{U}\ W\ Z)}$$

# Polymorphic return values 2/2

Looking at `if` / `cond` in functions we get

plus

```
(define (plus l r)
  (cond
    [(and (number? l) (number? r)) (+ l r)]
    [(and (string? l) (string? r)) (string-append l r)]
    [else (error 'plus "mismatch")]))
(test (plus 6 7) 13)
(test (plus "hello" " world") "hello world")
```

- what is the type of `plus`?
- why don't the parameters need annotation
- should we anyway?