Linear program Another term for linear programming model.

Linear functions Mathematical expressions in which the variables appear in separate terms and are raised to the first power.

Feasible solution A solution that satisfies all the constraints.

Feasible region The set of all feasible solutions.

Slack variable A variable added to the left-hand side of a less-than-or-equal-to constraint to convert the constraint into an equality. The value of this variable can usually be interpreted as the amount of unused resource.

Standard form A linear program in which all the constraints are written as equalities. The optimal solution of the standard form of a linear program is the same as the optimal solution of the original formulation of the linear program.

Redundant constraint A constraint that does not affect the feasible region. If a constraint is redundant, it can be removed from the problem without affecting the feasible region.

Extreme point Graphically speaking, extreme points are the feasible solution points occurring at the vertices or "corners" of the feasible region. With two-variable problems, extreme points are determined by the intersection of the constraint lines.

Surplus variable A variable subtracted from the left-hand side of a greater-than-or-equal-to constraint to convert the constraint into an equality. The value of this variable can usually be interpreted as the amount over and above some required minimum level.

Alternative optimal solutions The case in which more than one solution provides the optimal value for the objective function.

Infeasibility The situation in which no solution to the linear programming problem satisfies all the constraints.

Unbounded If the value of the solution may be made infinitely large in a maximization linear programming problem or infinitely small in a minimization problem without violating any of the constraints, the problem is said to be unbounded.

PROBLEMS

SELF test

1. Which of the following mathematical relationships could be found in a linear programming model, and which could not? For the relationships that are unacceptable for linear programs, state why.
   - a. $-1x_1 + 2x_2 - 1x_3 \le 70$
   - b. $2x_1 - 2x_3 = 50$
   - c. $1x_1 - 2x_2^2 + 4x_3 \le 10$
   - d. $3\sqrt{x_1} + 2x_2 - 1x_3 \ge 15$
   - e. $1x_1 + 1x_2 + 1x_3 = 6$
   - f. $2x_1 + 5x_2 + 1x_1x_2 \le 25$

SELF test

2. Find the feasible solution points for the following constraints:
   - a. $4x_1 + 2x_2 \le 16$
   - b. $4x_1 + 2x_2 \ge 16$
   - c. $4x_1 + 2x_2 = 16$
3. Show a separate graph of the constraint lines and feasible solutions for each of the following constraints:
   - a. $3x_1 + 2x_2 \le 18$
   - b. $12x_1 + 8x_2 \ge 480$
   - c. $5x_1 + 10x_2 = 200$

4. Show a separate graph of the constraint lines and feasible solutions for each of the following constraints:
   a. $3x_1 - 4x_2 \geq 60$
   b. $-6x_1 + 5x_2 \leq 60$
   c. $5x_1 - 2x_2 \leq 0$
5. Show a separate graph of the constraint lines and feasible solutions for each of the following constraints:
   a. $x_1 \geq 0.25\,(x_1 + x_2)$
   b. $x_2 \leq 0.10\,(x_1 + x_2)$
   c. $x_1 \leq 0.50\,(x_1 + x_2)$

SELF test

6. Three objective functions for linear programming problems are $7x_1 + 10x_2$, $6x_1 + 4x_2$, and $-4x_1 + 7x_2$. Determine the slope of each objective function. Show the graph of each for objective function values equal to 420.

SELF test

7. Identify the feasible region for the following set of constraints:

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 &\geq 30 \\ 1x_1 + 5x_2 &\geq 250 \\ \tfrac{1}{4}x_1 + \tfrac{1}{2}x_2 &\leq 50 \\ x_1, x_2 &\geq 0 \end{aligned}$$

8. Identify the feasible region for the following set of constraints:

$$\begin{aligned} 2x_1 - 1x_2 &\leq 0 \\ -1x_1 + 1.5x_2 &\leq 200 \\ x_1, x_2 &\geq 0 \end{aligned}$$

9. Identify the feasible region for the following set of constraints:

$$\begin{aligned} 3x_1 - 2x_2 &\geq 0 \\ 2x_1 - 1x_2 &\leq 200 \\ 1x_1 \qquad &\leq 150 \\ x_1, x_2 &\geq 0 \end{aligned}$$

SELF test

10. For the linear program

$$\begin{aligned} \text{Max} \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & \\ & 1x_1 + 2x_2 \leq 6 \\ & 5x_1 + 3x_2 \leq 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

find the optimal solution using the graphical solution procedure. What is the value of the objective function at the optimal solution?

11. Solve the following linear program using the graphical solution procedure.

$$\begin{aligned} \text{Max} \quad & 5x_1 + 5x_2 \\ \text{s.t.} \quad & \\ & 1x_1 \qquad \leq 100 \\ & \qquad 1x_2 \leq 80 \\ & 2x_1 + 4x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

12. Consider the following linear programming model:

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 3x_2 \\
\text{s.t.} \quad & \\
& 2x_1 + 4x_2 \le 12 \\
& 6x_1 + 4x_2 \le 24 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

a. Find the optimal solution using the graphical solution procedure.
b. If the objective function is changed to $2x_1 + 6x_2$, what will the optimal solution be?
c. How many extreme points are there? What are the values of x_1 and x_2 at each extreme point?

13. Consider the following linear program:

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 \\
\text{s.t.} \quad & \\
& 2x_1 + 2x_2 \le 8 \\
& 3x_1 + 2x_2 \le 12 \\
& 1x_1 + 0.5x_2 \le 3 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

a. Find the optimal solution using the graphical solution procedure. What is the value of the objective function?
b. Does this linear program have a redundant constraint? If so, what is it? Does the solution change if the redundant constraint is removed from the model? Explain.

SELF test

14. Consider the following linear program:

$$
\begin{aligned}
\text{Max} \quad & 1x_1 + 2x_2 \\
\text{s.t.} \quad & \\
& 1x_1 \le 5 \\
& 1x_2 \le 4 \\
& 2x_1 + 2x_2 = 12 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

a. Show the feasible region.
b. What are the extreme points of the feasible region?
c. Find the optimal solution using the graphical procedure.

15. Refer to the Par, Inc., problem described in Section 2.1. Suppose that Par's management encounters each of the following situations:
a. The accounting department revises its estimate of the profit contribution for the deluxe bag to \$18 per bag.
b. A new low-cost material is available for the standard bag, and the profit contribution per standard bag can be increased to \$20 per bag. (Assume the profit contribution of the deluxe bag is the original \$9 value.)
c. New sewing equipment is available that would increase the sewing operation capacity to 750 hours. (Assume $10S + 9D$ is the appropriate objective function.)

If each of these conditions is encountered separately, what are the optimal solution and the total profit contribution for each situation?

16. Refer to the feasible region for the Par, Inc., problem in Figure 2.13.
    a. Develop an objective function that will make extreme point (5) the optimal extreme point.
    b. What is the optimal solution using the objective function you selected in part (a)?
    c. What are the values of the slack variables associated with this solution?

SELF test

17. Write the following linear program in standard form:

$$\begin{aligned}
\text{Max} \quad & 5x_1 + 2x_2 + 8x_3 \\
\text{s.t.} \quad & \\
& 1x_1 + 2x_2 + \tfrac{1}{2}x_3 \le 420 \\
& 2x_1 + 3x_2 - 1x_3 \le 610 \\
& 6x_1 - 1x_2 + 3x_3 \le 125 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

18. For the linear program

$$\begin{aligned}
\text{Max} \quad & 4x_1 + 1x_2 \\
\text{s.t.} \quad & \\
& 10x_1 + 2x_2 \le 30 \\
& 3x_1 + 2x_2 \le 12 \\
& 2x_1 + 2x_2 \le 10 \\
& x_1, x_2 \ge 0
\end{aligned}$$

    a. Write this linear program in standard form.
    b. Find the optimal solution using the graphical solution procedure.
    c. What are the values of the three slack variables at the optimal solution?

19. Given the linear program

$$\begin{aligned}
\text{Max} \quad & 3x_1 + 4x_2 \\
\text{s.t.} \quad & \\
& -1x_1 + 2x_2 \le 8 \\
& 1x_1 + 2x_2 \le 12 \\
& 2x_1 + 1x_2 \le 16 \\
& x_1, x_2 \ge 0
\end{aligned}$$

    a. Write the linear program in standard form.
    b. Find the optimal solution using the graphical solution procedure.
    c. What are the values of the three slack variables at the optimal solution?

20. Embassy Motorcycles (EM) manufactures two lightweight motorcycles designed for easy handling and safety. The EZ-Rider model has a new engine and a low profile that make it easy to balance. The Lady-Sport model is slightly larger, uses a more traditional engine, and is specifically designed to appeal to women riders. Embassy produces the engines for both models at its Des Moines, Iowa, plant. Each EZ-Rider engine requires 6 hours of manufacturing time and each Lady-Sport engine requires 3 hours of manufacturing time. The Des Moines plant has 2100 hours of engine manufacturing time available for the next production period. Embassy's motorcycle frame supplier can supply as many EZ-Rider frames as needed. However, the Lady-Sport frame is more complex and the supplier can provide only up to 280 Lady-Sport frames for the next production period. Final assembly and testing requires 2 hours for each EZ-Rider model and 2.5 hours for each Lady-Sport

model. A maximum of 1000 hours of assembly and testing time are available for the next production period. The company's accounting department projects a profit contribution of \$2400 for each EZ-Rider produced and \$1800 for each Lady-Sport produced.

a. Formulate a linear programming model that can be used to determine the number of units of each model that should be produced in order to maximize the total contribution to profit.
b. Find the optimal solution using the graphical solution procedure.
c. Which constraints are binding?

21. RMC, Inc., is a small firm that produces a variety of chemical products. In a particular production process, three raw materials are blended (mixed together) to produce two products: a fuel additive and a solvent base. Each ton of fuel additive is a mixture of $\frac{2}{5}$ ton of material 1 and $\frac{3}{5}$ of material 3. A ton of solvent base is a mixture of $\frac{1}{2}$ ton of material 1, $\frac{1}{5}$ ton of material 2, and $\frac{3}{10}$ ton of material 3. After deducting relevant costs, the profit contribution is \$40 for every ton of fuel additive produced and \$30 for every ton of solvent base produced.

RMC's production is constrained by a limited availability of the three raw materials. For the current production period, RMC has available the following quantities of each raw material:

Raw Material	Amount Available for Production
Material 1	20 tons
Material 2	5 tons
Material 3	21 tons

Assuming that RMC is interested in maximizing the total profit contribution, answer the following:

a. What is the linear programming model for this problem?
b. Find the optimal solution using the graphical solution procedure. How many tons of each product should be produced, and what is the projected total profit contribution?
c. Is there any unused material? If so, how much?
d. Are there any redundant constraints? If so, which ones?

SELF test

22. Kelson Sporting Equipment, Inc., makes two different types of baseball gloves: a regular model and a catcher's model. The firm has 900 hours of production time available in its cutting and sewing department, 300 hours available in its finishing department, and 100 hours available in its packaging and shipping department. The production time requirements and the profit contribution per glove are given in the following table.

	Production Time (hours)			
Model	Cutting and Sewing	Finishing	Packaging and Shipping	Profit/Glove
Regular model	1	$\frac{1}{2}$	$\frac{1}{8}$	\$5
Catcher's model	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	\$8

Assuming that the company is interested in maximizing the total profit contribution, answer the following:

a. What is the linear programming model for this problem?
b. Find the optimal solution using the graphical solution procedure. How many gloves of each model should Kelson manufacture?
c. What is the total profit contribution Kelson can earn with the listed production quantities?