Realizability Problems for Convex Polytopes (and Relatives)

or

Excursions in coordinate-free convex geometry

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Overview

1. Polytopes and Linear Programming
2. Constructing Polytopes with Long Paths
3. Abstract Point Configurations 1: Chirotopes
4. Searching For Chirotopes
5. Abstract Point Conf. 2: Hyperline Sequences
6. Conclusions

1. Polytopes and Linear Programming

Linear Programming

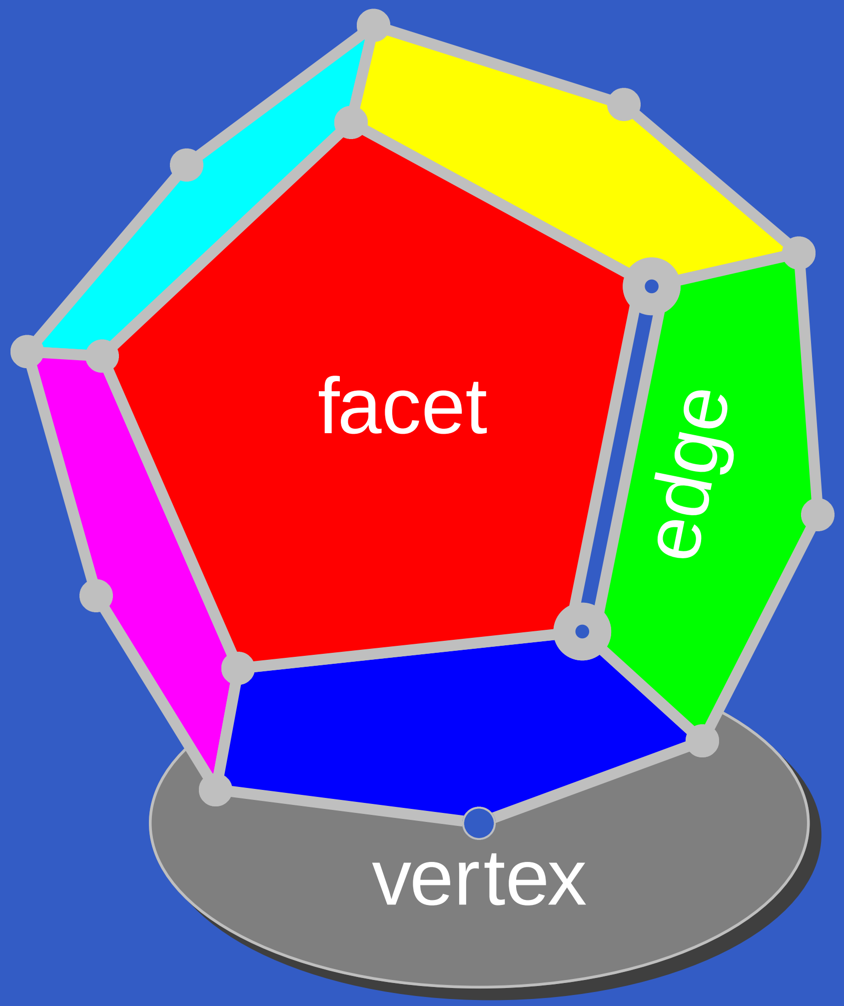
minimize $\quad c \cdot x$

Such that

$$Ax \leq b$$

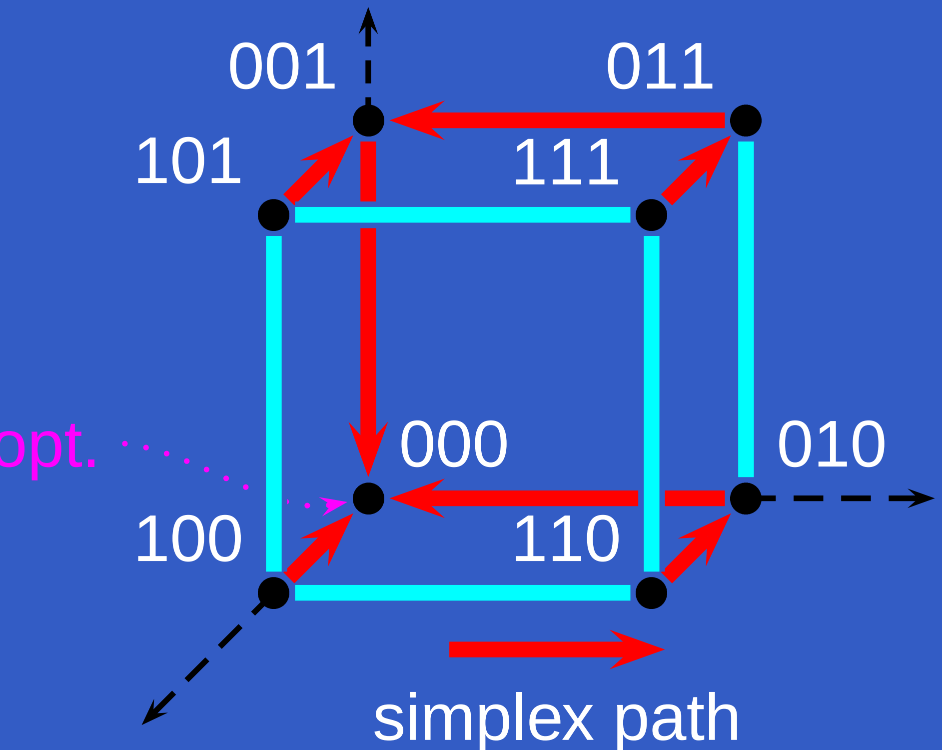
- $P = \{ x \mid Ax \leq b \}$ is called a (convex) *polyhedron*
- Bounded polyhedra are called (convex) *polytopes*.

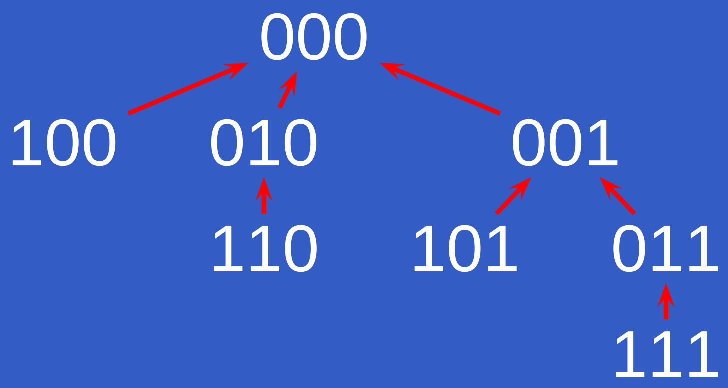
Polytopes



- Face: $\cap$ with supporting hyperplane
- $\mathrm{conv}(X) = \{\, \lambda X \mid \lambda \geq 0, \sum_i \lambda_i = 1 \,\}$.
- $P = \mathrm{conv}(\mathrm{vertices}(P))$

The Simplex Method





$d(u, v) \equiv$ length of the shortest edge-path from u to v.

diameter $\equiv \max_{(u,v)} d(u, v)$

The Hirsch Conjecture

Conjecture (Hirsch, 1957)
Any polytope defined by n inequalities in d dimensions has diameter at most $n - d$.

Theorem (Kalai, 1992)
Any polytope defined by n inequalities in d dimensions has diameter at most

$$2(2d)^{\log_2(n)}\ .$$

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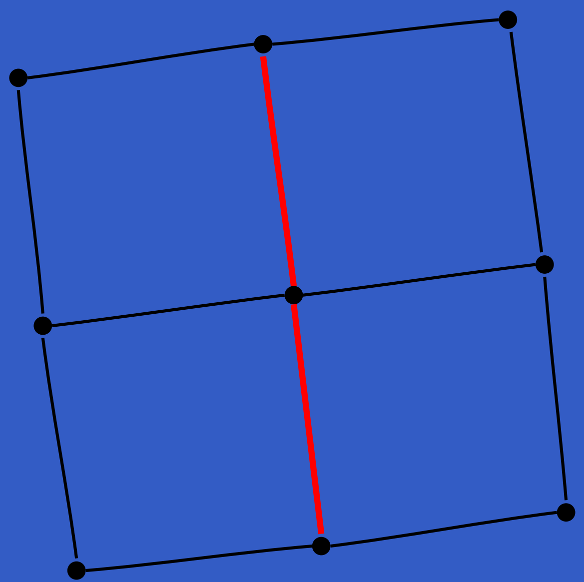
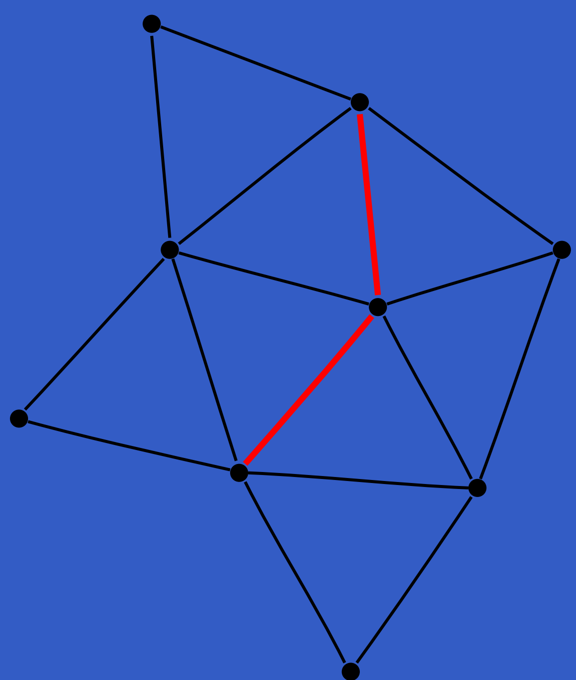
2. From Paths to Polytopes

The Grand Program

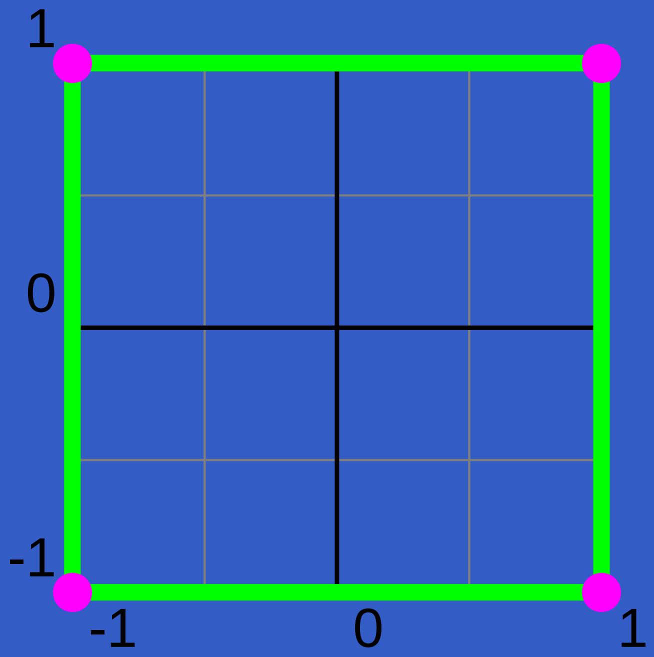
Idea For each *combinatorially distinct* long path, try to build a polytope out of it.

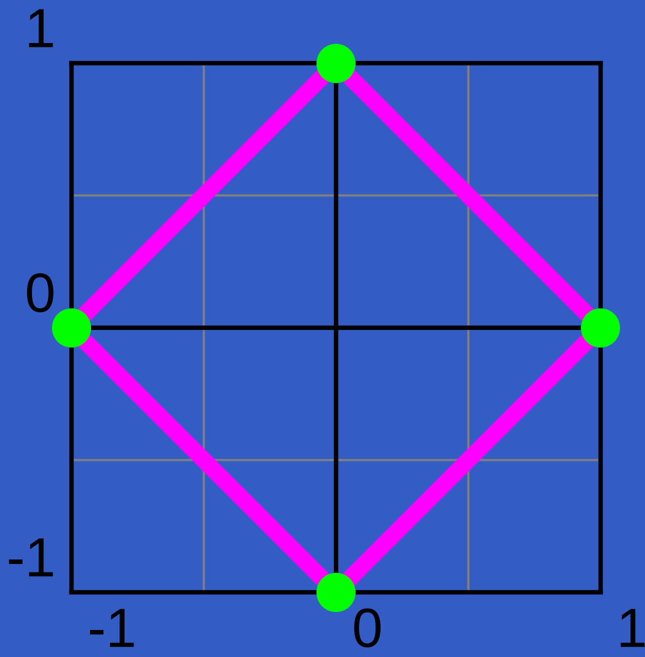
Problem One path pretty much looks like the next.





Polarity: Paths to Path Complexes



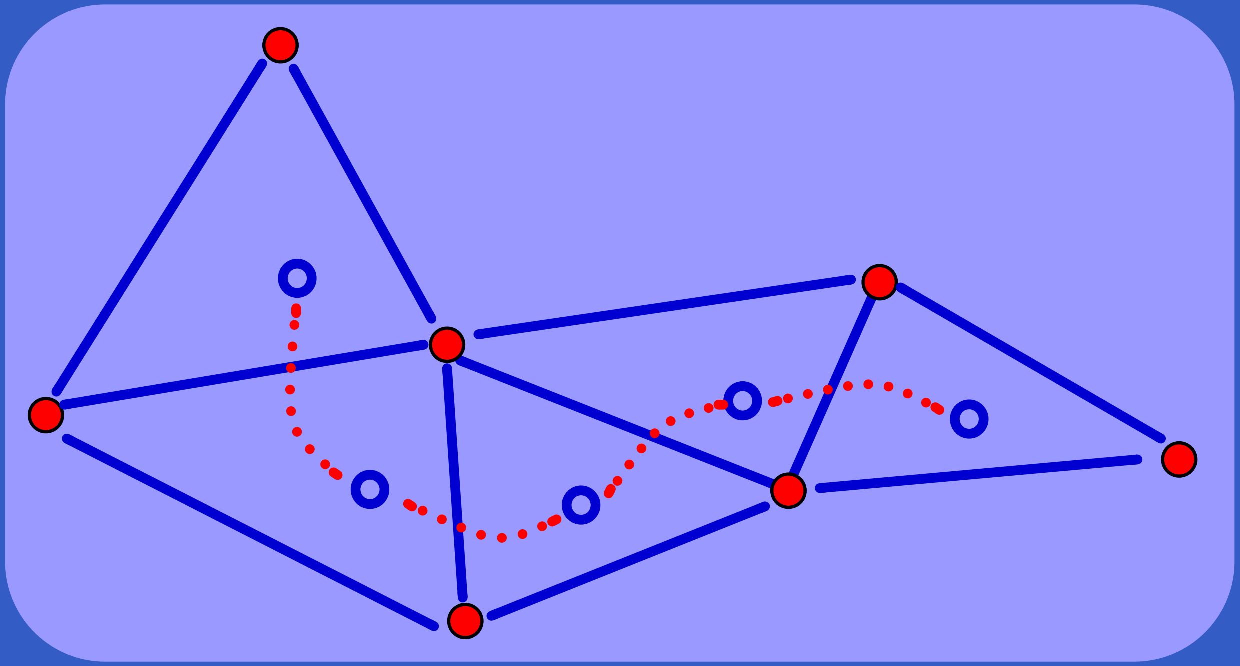


- *polar:* $P^* = \mathrm{conv}\{\, y \mid \forall x \in P, y \cdot x \leq 1 \,\}$
- vertices $\leftrightarrow$ facets, inclusion inverted.

Path Complexes

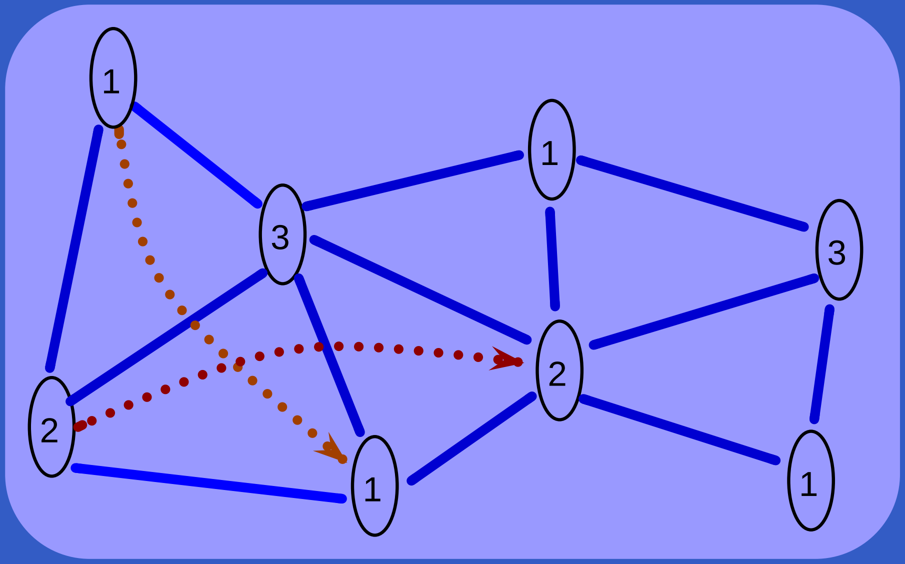
Simplicial Complex Family of d-subsets of $\{1 \ldots n\}$

Path Complex Simplicial complex whose dual graph is a path.



Enumerating Path Complexes [BBHK]

Non Revisiting Paths



label sequence: 12131

- Each pivot introduces a new vertex.
- Label first facet in order of departure. Labels follow pivots.

Label Sequences

Directed Paths Label sequences $\langle s_j \rangle$ such that

- $s_j \neq s_{j-1}$, and
- If $a < b$, a occurs before b.

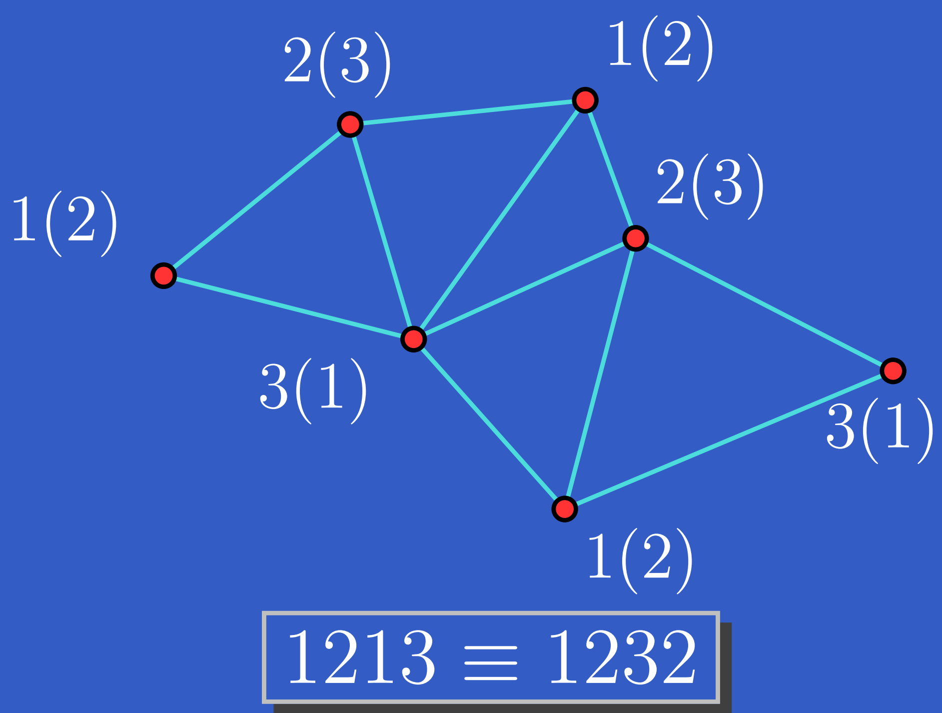
End Disjoint Paths (Restricted Growth Functions)

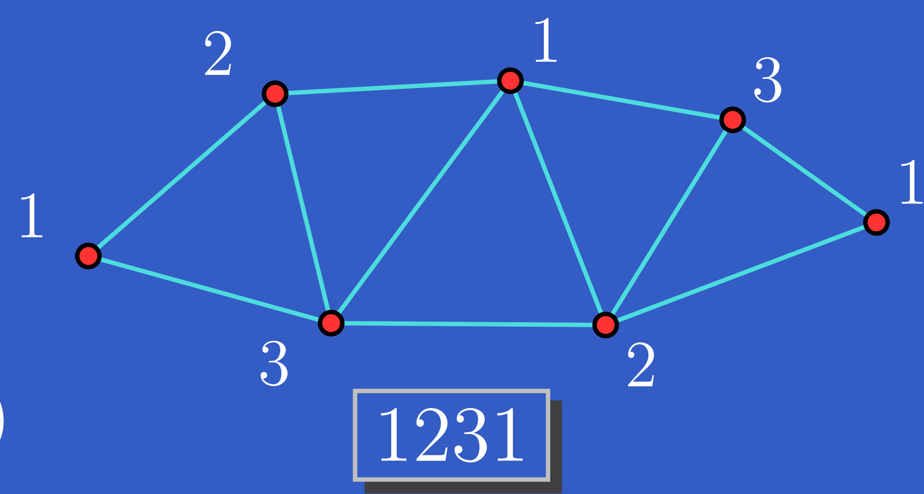
$$\max_j s_j = d\,.$$

$$t(d,l) \equiv \#\text{e.d.d. } (d,l)\text{-paths},$$

$$t(d,l) = \left\{ \begin{matrix} l-1 \\ d-1 \end{matrix} \right\}$$

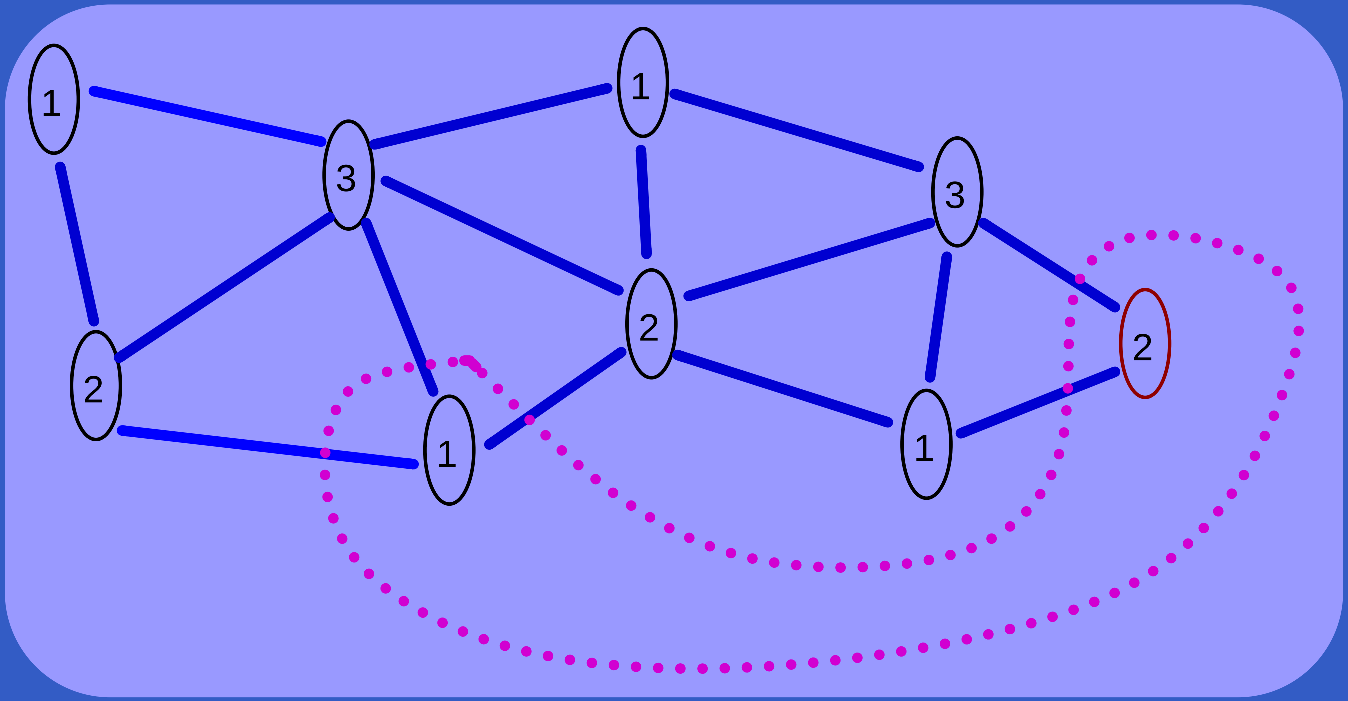
Symmetric Paths





- *symmetric* $\equiv$ same label seq. from both ends.
- #unlabelled paths $= (t(d,l) + s(d,l))/2$

Revisiting Paths



- Model *revisits* by identifying pairs of vertices
- Characterization of 0 and 1 revisit paths in [BBHK]

3. Chirotopes: Abstract Point Configurations

Searching for polytopes

- Nominally, an n-vertex d-polytope is a point in $\mathbb{R}^{nd}$.

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- The pieces corresponding to a polytope are *not* nice:

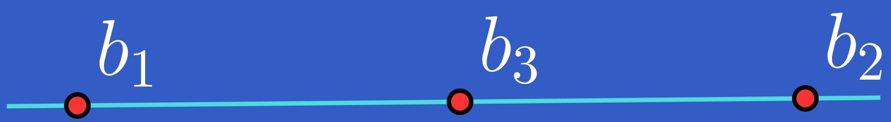
  The realization spaces of polytopes are equivalent to the solutions of arbitrary sets of polynomial inequalities (Richter-Gebert,Mnëv).

Encoding Point Sets

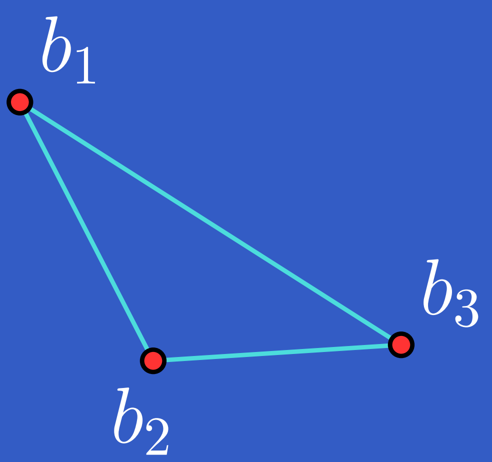
basis $\equiv B \subset \mathbb{R}^{(d+1)\times d}$

$$\chi(B) = \operatorname{sign}\det \begin{bmatrix} B & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \end{bmatrix}$$

Idea: Which side of the hyperplane defined by $\{\, b_1 \ldots b_d \,\}$ is b_{d+1} on.



$\chi(B) = 0$



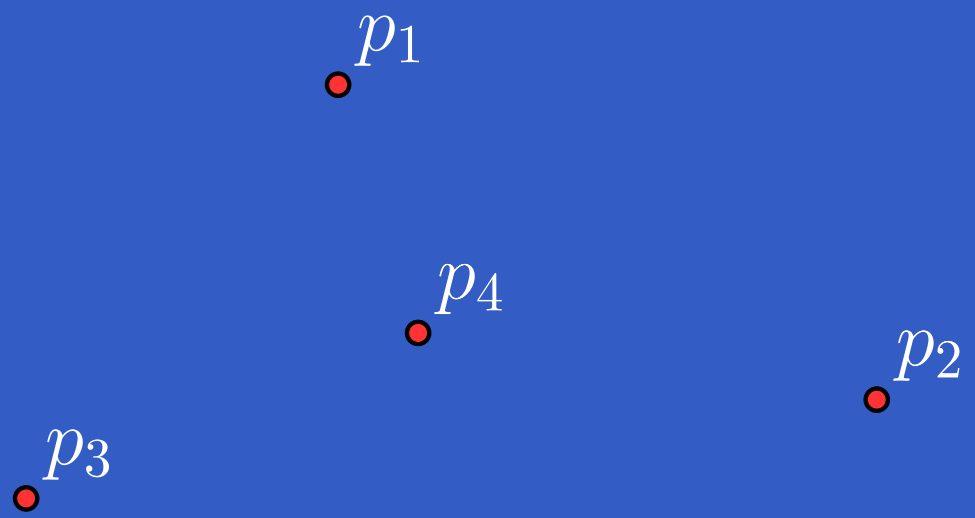
$\chi(B) = +1 =$ ccw

(Realizable) Chirotopes

The *chirotope* χ of $P \subset \mathbb{R}^d$ is the map

$$B \in P^{d+1} \to \chi(B) \in \{\, 0, \pm 1 \,\}$$

$$[i_1, i_2, \ldots, i_{d+1}] \equiv \chi(\{\, p_{i_1}, \ldots p_{i_{d+1}} \,\})$$



$$[1, 2, 3] = -1$$

$$[1, 2, 4] = -1$$

$$[1, 3, 4] = +1$$

$$[2, 3, 4] = -1$$

Alternating Sign Maps

Given $N = \{1 \ldots n\}$, a rank $r = d + 1$,
$\chi : N^r \to \{-1, 0, +1\}$ is

alternating if
$$[b_1 \ldots i \ldots j \ldots b_r] = -1 \cdot [b_1 \ldots j \ldots i \ldots b_r]$$
(determinant w.r.t. row swap).

uniform if $\forall B \; \chi(B) \neq 0$.
(Non-degeneracy)

(Combinatorial) Chirotopes

A uniform alternating sign map χ is a (uniform) *chirotope* if

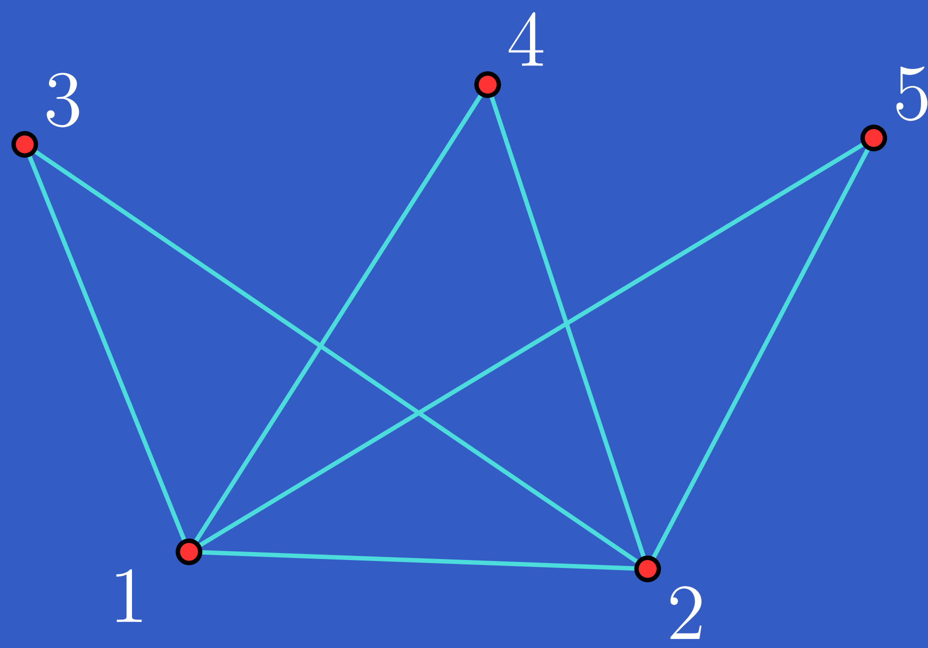
$$\begin{aligned} \forall \lambda \in N^{r-2}, \forall \{\, a, b, c, d \,\} \subset N \setminus \lambda, \\ \{\, +1, -1 \,\} \subset \{[\lambda\ a\ b] \cdot [\lambda\ c\ d], \\ -[\lambda\ a\ c] \cdot [\lambda\ b\ d], \\ [\lambda\ a\ d] \cdot [\lambda\ b\ c]\} \end{aligned}$$

(3-term Plücker-Graßmann Identity)

Matroid Polytopes

Given a uniform chirotope (N, χ):

- $F \subset N^d$ is a *facet* if $\forall \{ j, k \} \subset N \setminus F$, $[F\ j] = [F\ k]$.
- (N, χ) is a *matroid polytope* if $\forall x \in N$, x is contained in some facet.



$$[1\ 2\ 3] = [1\ 2\ 4] = [1\ 2\ 5]$$

Matroid Polytope Completion

Partial Chirotope A map $\chi(B) : N^r \to \{-1, +1, ?\}$

Matroid Polytope Completion

Given: A partial chirotope (N, χ), and some subset $\mathcal{F}$ of the facets.

Question: Is there a chirotope (N, χ^*) consistent with χ such that each $F \in \mathcal{F}$ is a facet of χ^*.

Complications

Geodesic Embedding Given $s, t \in \mathcal{F}$, add constraint $d(s,t) = k$.

MPC is NP-Hard in rank 3 with $\mathcal{F} = \emptyset$. Tschirnitz CCCG2001

Realizability is also NP-Hard (Richter-Gebert 1995, Mnëv). Non-realizable instances for $d = 3$, $n = 10$ and $d = 4$, $n = 9$

4. Direct Approaches to Chirotope Completion

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Approach I: Backtracking

- `oms`: Backtracking algorithm to find "satisfying" basis signs

1. Choose a sign
2. Find the consequences
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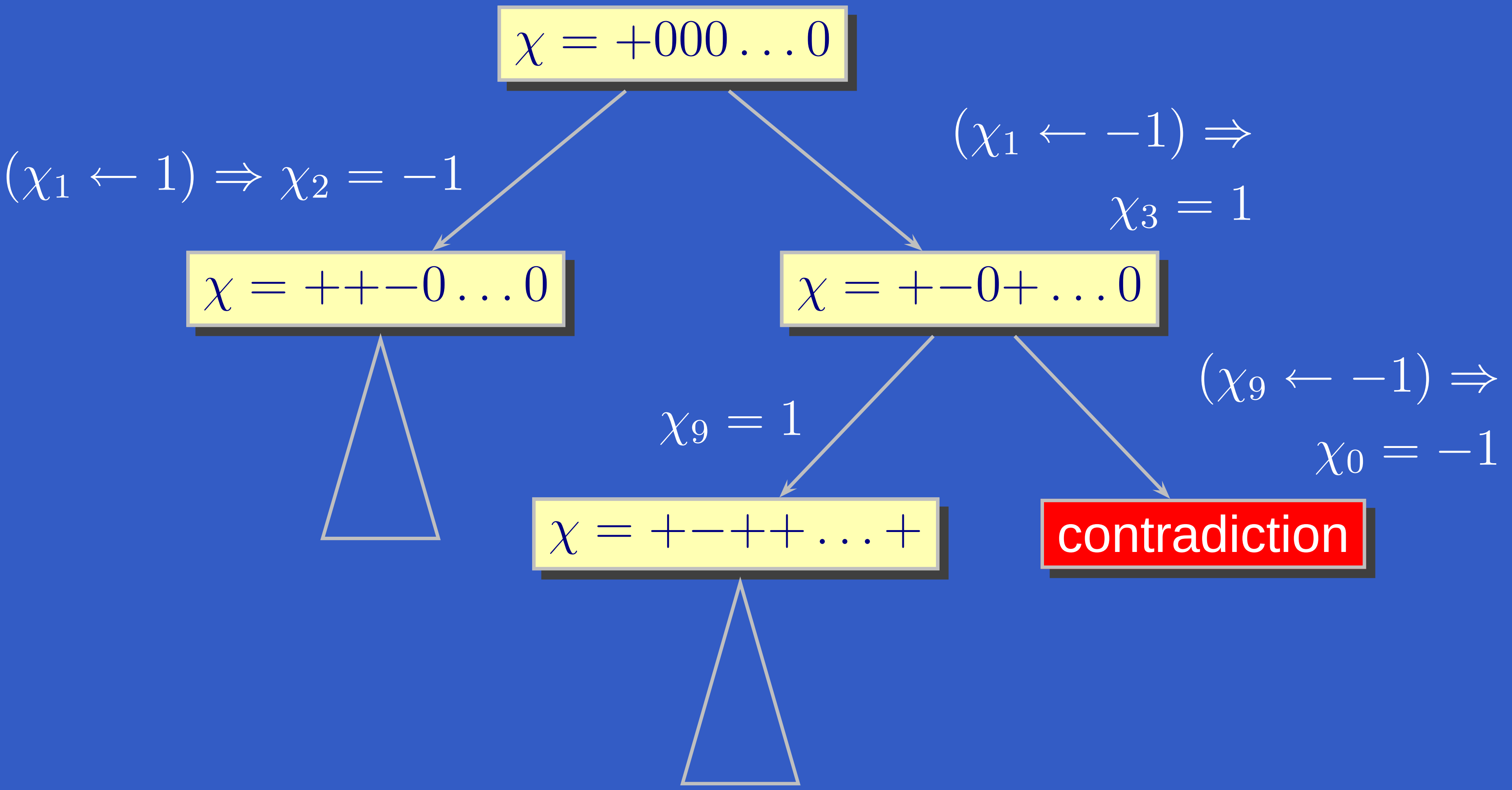
- `oms`: Backtracking algorithm to find "satisfying" basis signs
  1. Choose a sign
  2. Find the consequences
  3. (Maybe) recurse
- 3 sets of constraints: Plücker, boundary, distance
- Analogous to Davis-Putnam SAT Procedure
  - Singleton clause $\equiv$ forced variable

Backtracking Tree



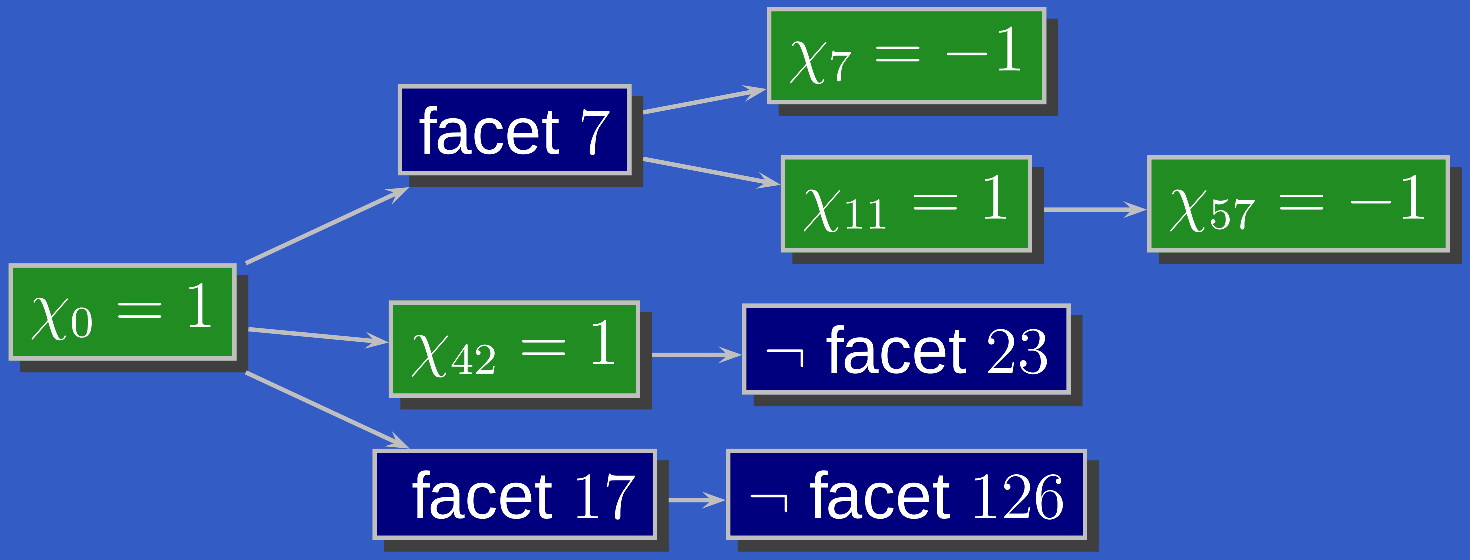
Forcing Variables

$$\{ -1, +1 \} \subset \{ x_1 \cdot x_2, -(x_3 \cdot x_4), x_5 \cdot x_6 \} \quad \text{(Plücker)}$$

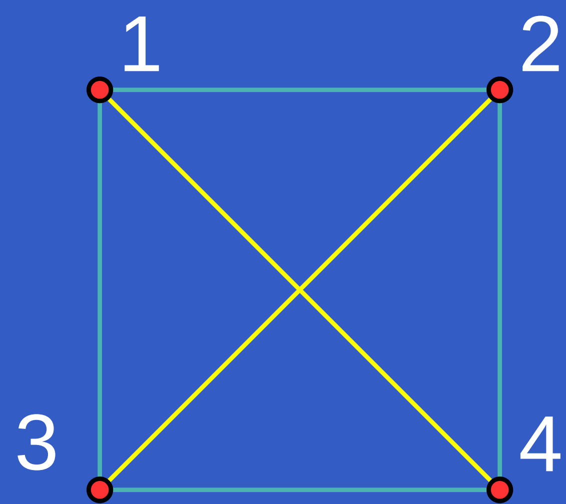
$$x_1 = x_2 = x_3 \cdots = x_{n-d} \quad \text{(On Boundary)}$$

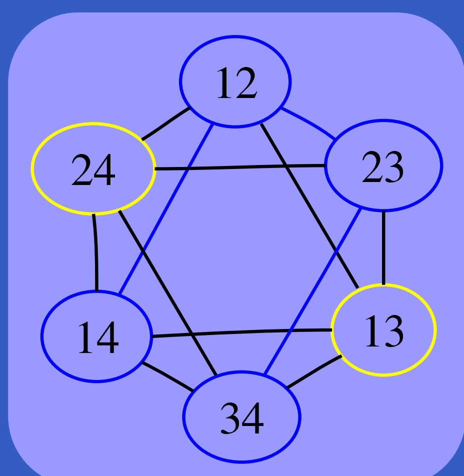
$$\{ -1, +1 \} \subset \{ x_1, x_2, \ldots x_{n-d} \} \quad \text{(Off Boundary)}$$

$$\neg \mathcal{F}[i] \vee \neg \mathcal{F}[j] \vee \neg \mathcal{F}[k] \ldots \quad \text{(Diameter)}$$

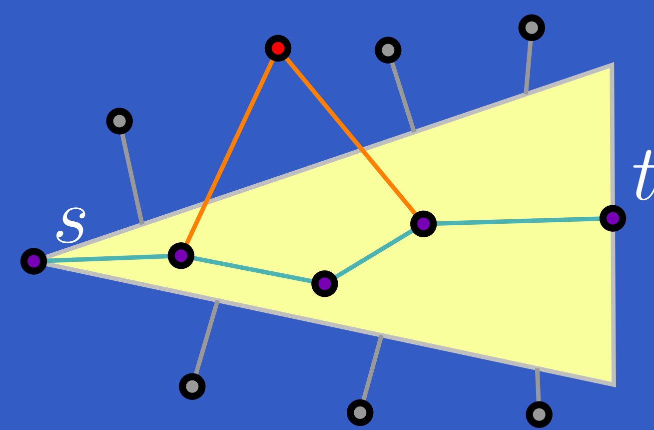


Keeping your distance





Pivot Graph



- Maintain *fringed* shortest path tree(s)
- Forbid short cuts

Approach II: $(0,1)$-LP

Plücker Take convex hull of valid $(0,1)$ points in $\mathbb{R}^6$. Lift 16 inequalities to $\mathbb{R}^{\binom{n}{d+1}}$.

For $d = 4$, $n = 11$, roughly 160000 inequalities in 410 binary variables, 10 nonzeros per row.

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Diameter Forbid all *possible* short paths.

- Enumerate paths in pivot graph.
- Generate 2 inequalities for each path.

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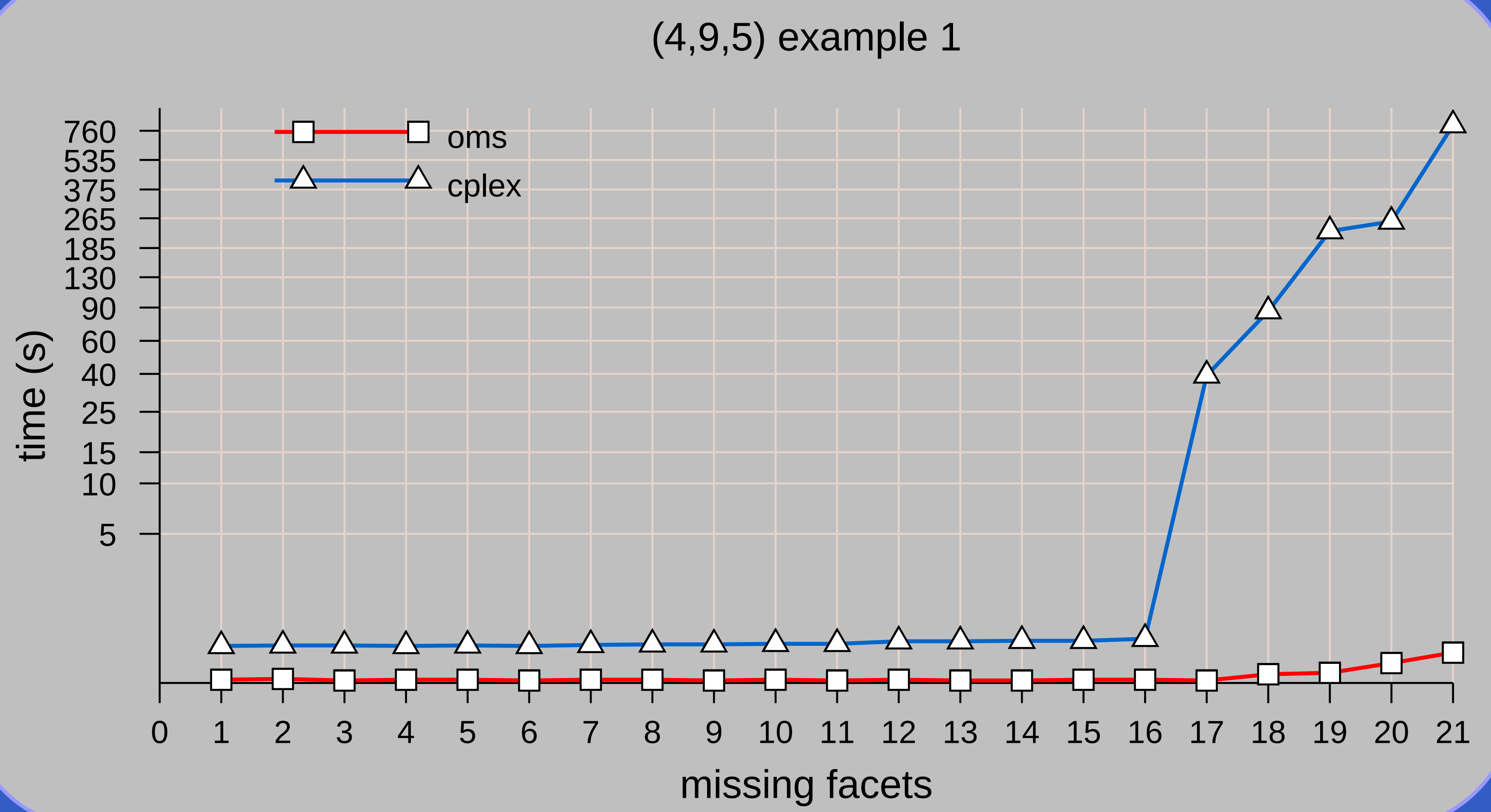
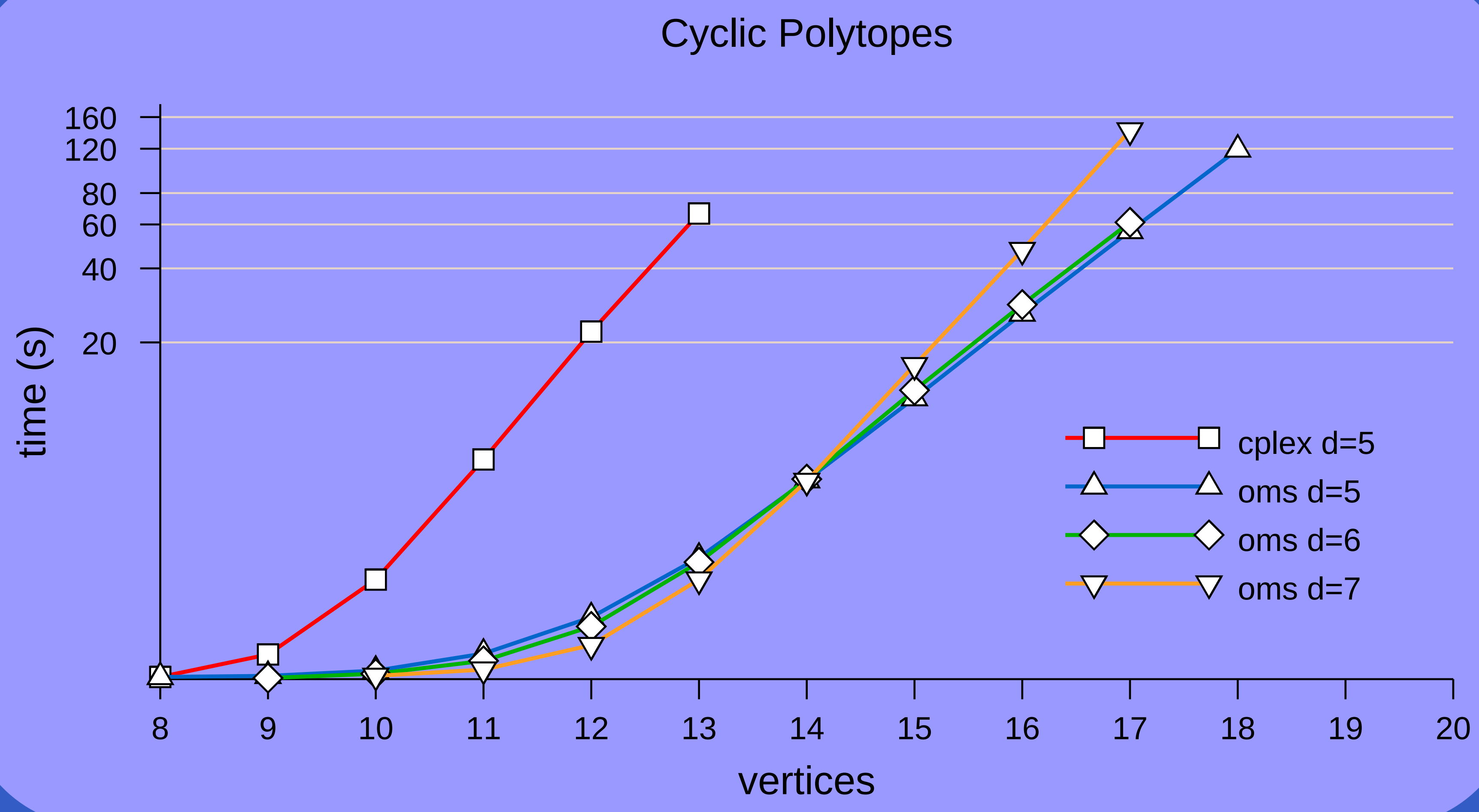
oms vs. cplex (I)



oms vs. cplex (II)

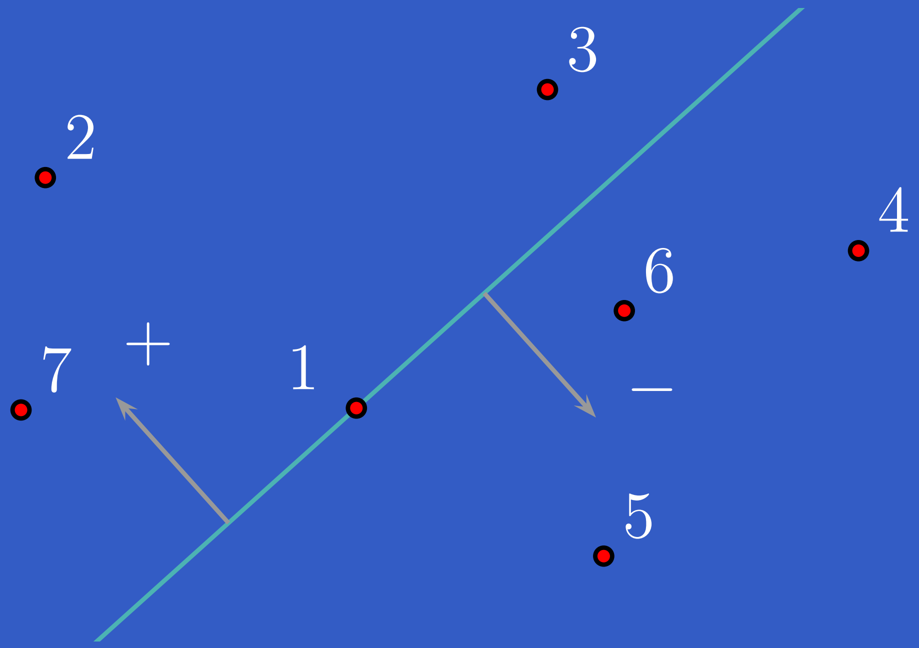
oms vs. cplex (III)



5. Incremental Construction: Hyperline Sequences

Sweeping Around a Hyperline

- Sweep hyperplane around $d - 1$ points.
- Record the (cyclic) order points are reached.



$$\mathrm{hls}(1) = (-6, -4, +7, -5, +2, +3)$$

Hyperline Sequences

- $N = \{1 \ldots n\}$.
- A *hyperline sequence* of $\lambda \in N^{d-1}$,

$$\mathrm{hls}(\lambda) = (\sigma_1\mu_1 \ \sigma_2\mu_2 \ \ldots \ \sigma_{n-d+1}\mu_{n-d+1})$$

Where

$$\sigma \in \{\,+1, -1\,\}^{n-d+1}$$
$$\mu \in \mathrm{permutations}(N \setminus \lambda)$$

Hyperline Configurations

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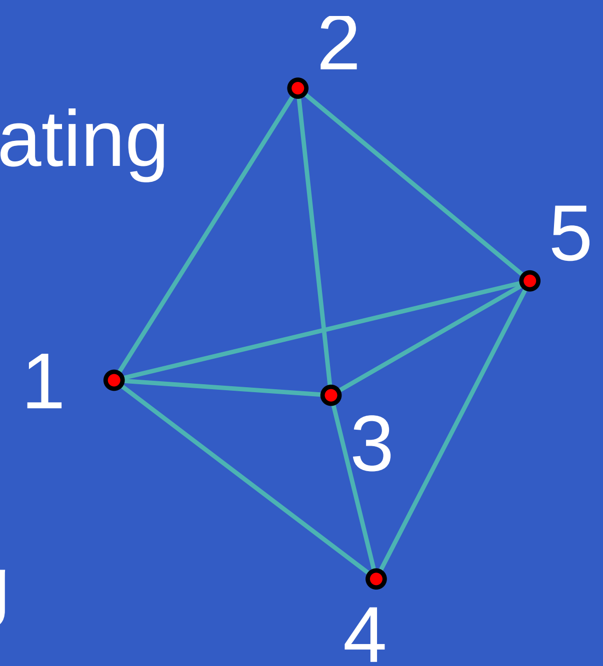
- Plücker equations are implicit.
- Incremental algorithm (with backtracking) due to Bokowski and Guedes de Oliviera tests for flat embedding.

Sign Alternation

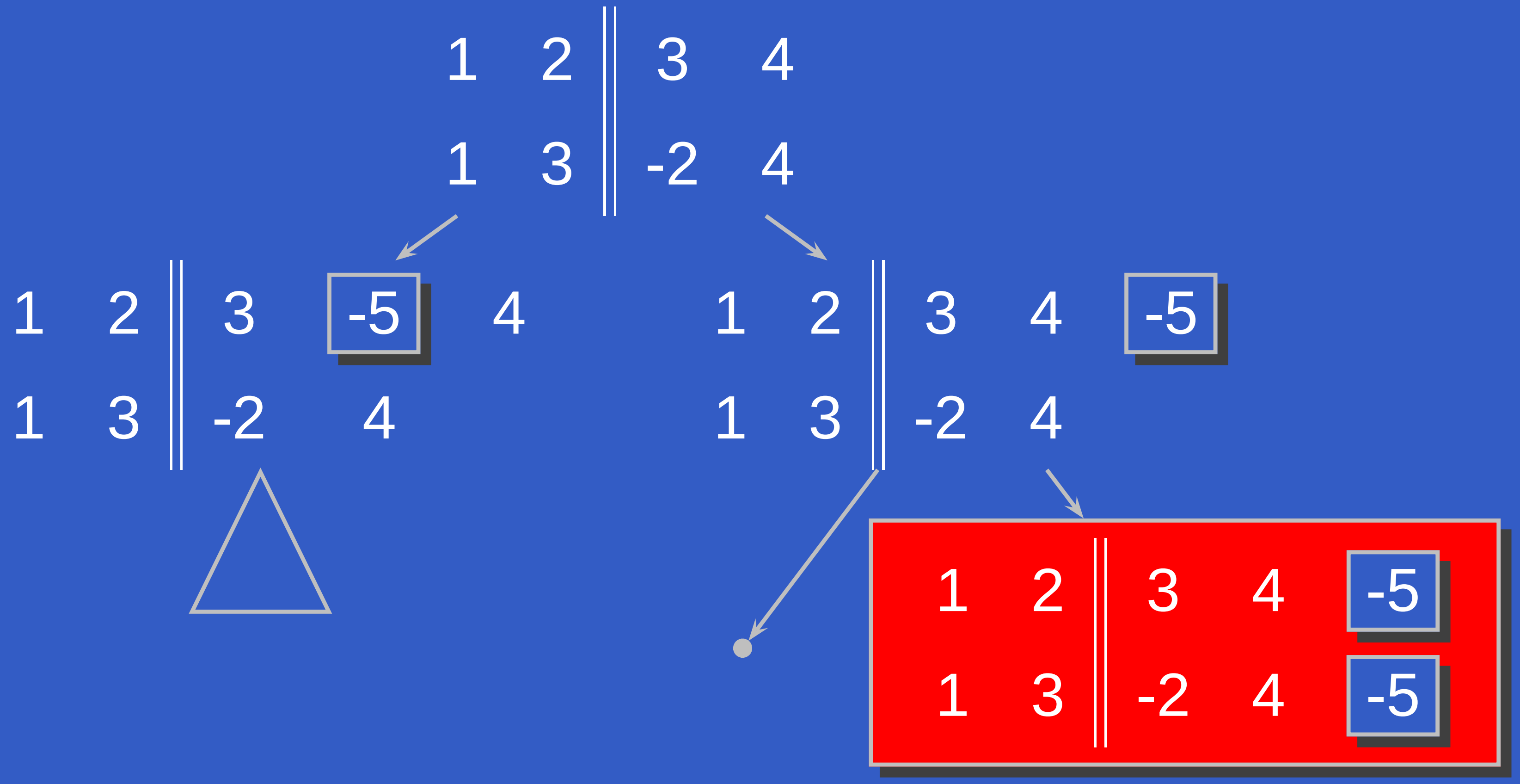
1	2	3		4	5	non-alternating
1	3	-2		4	5	
1	4	-2		-3	5	
$\vdots$						
2	4	1		-3	5	alternating



Proposition

A hyperline is on the boundary if and only if it has a non-alternating sign sequence.

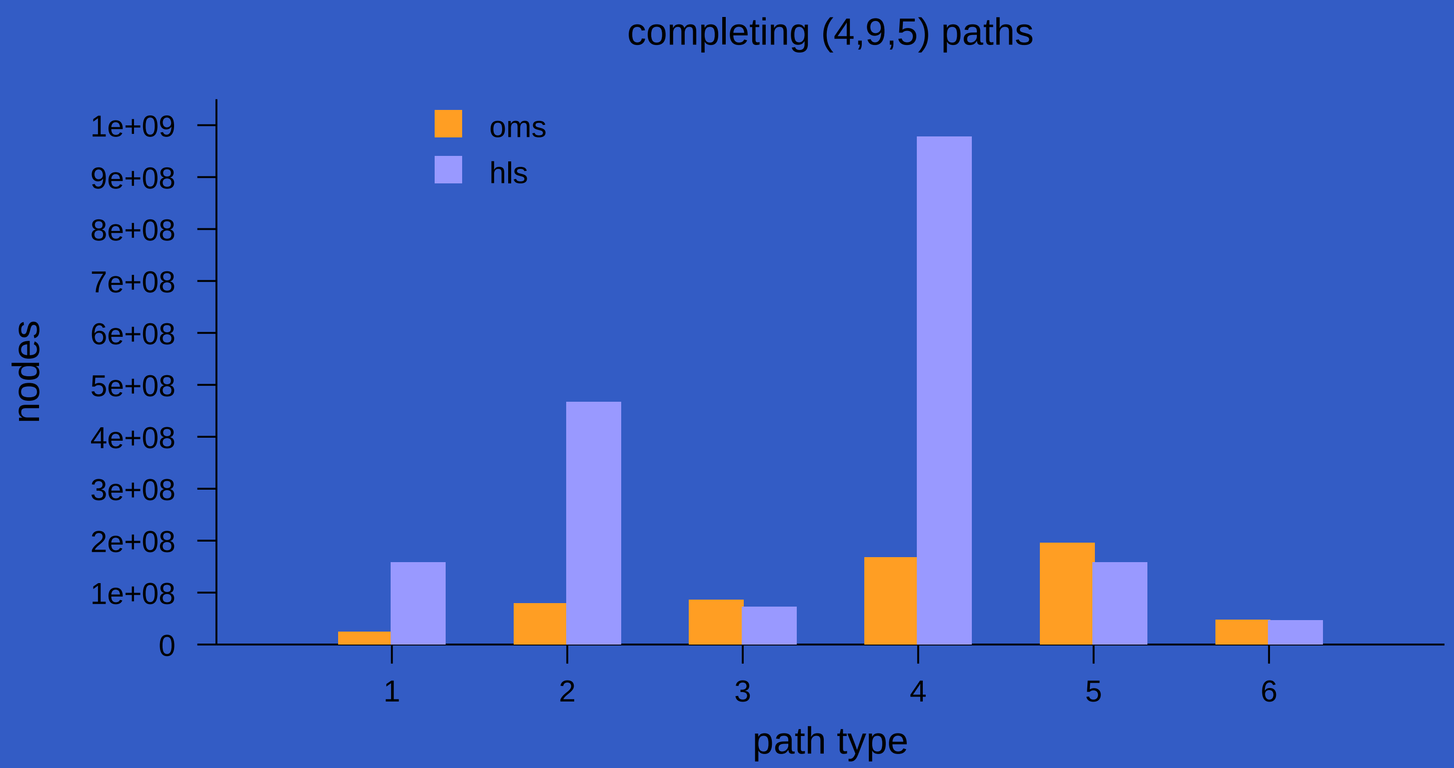
Incremental Construction



choose gap, update intervals, check constr., recurse

Hyperline versus Chirotope Search

Hyperline search algorithm, modified version of [BGdO], uses alternation test.



6. Concluding Remarks

Conclusions

- So far, no presented approach can solve $(4, 11)$ examples from only a path.
- Given most of the boundary, moderate sized problems can be tackled.
- Memory is the main limitation.
- Specialized backtracking solver seems competitive with (a) commercial ILP solver and incremental construction.

Remarks: Background

- Working directly with boundary harder?:
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  - Computational Experience of Holt.
- Simplicial polytopes $\subset$ UMP $\subset$ simplicial spheres
- Hirsch conjecture is false for spheres ($d = 12$, Manni)
- Both hyperline configurations and chirotopes are axiomatizations of *oriented matroids*

Remarks: Software

- Web test-drive available via *anonpbs* `http://lids.cs.unb.ca/online`
- `oms` has been parallelized using Marzetta's ZRAM toolkit. Speedup is about 85%. Further improvements possible