

# CS3383 Unit 0: Deeper into Asymptotics

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# Outline

## New asymptotic classes

Big-theta

little-o

little-omega

## Exponentials and logs

Two examples

The limit of ratios approach

# big- $\Theta$ example

$$\frac{n^2}{2} - 2n \in \Theta(n^2)$$

- ▶  $O$  clear
- ▶  $\Omega$ : common factors, choose  $c$  as an arbitrary constant at most  $1/2$

# little-o Example 1

$$f(n) \in o(g(n))$$

$$\forall c > 0 \exists n_0 \quad \forall n > n_0 \quad 0 \leq f(n) < cg(n)$$

$$2n^2 \in o(n^3)$$

► find  $n_0$  for arbitrary (unknown)  $c$ .

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$$2n^2 \notin o(n^2)$$

big-O solve for c  
little-o lower-bound  
c

# little- $\omega$ and big- $\Omega$

$$f(n) \in \Omega(g(n))$$

$$\exists c \exists n_0 \quad \forall n > n_0 \quad 0 \leq cg(n) \leq f(n)$$



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# little- $\omega$ implies big- $\Omega$

Definition ( $f(n) \in \omega(g(n))$ )

$$\forall c > 0 \exists n_0 \forall n > n_0 \quad 0 \leq cg(n) < f(n)$$

► assume little- $\omega$ , prove big- $\Omega$

# little- $\omega$ example

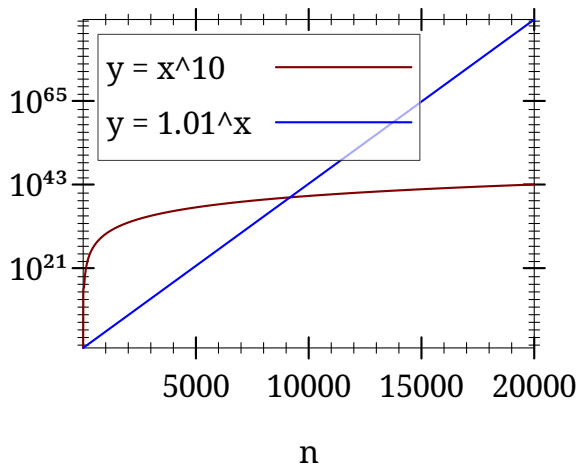
$$f(n) \in \omega(g(n))$$

$$\forall c > 0 \exists n_0 \forall n > n_0 \quad 0 \leq cg(n) < f(n)$$

$$n^3 \in \omega\left(\frac{n^2}{2} + 2n\right)$$

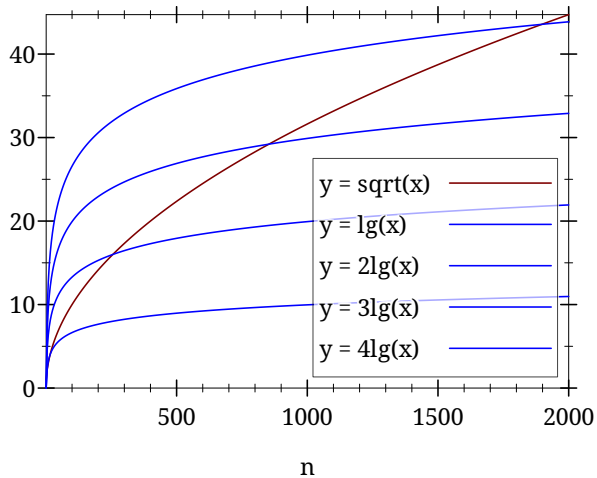
- ▶ find  $n^*$  s.t.  $n^2/2 > 2n$ ,  $\forall n \geq n^*$
- ▶ find  $n_0 \geq n^*$  s.t.  $n^3 > cn^2$ ,  $\forall n \geq n_0$

# Exponential versus Polynomial



$$(1.01)^n \in \omega(n^{10}) \subseteq \Omega(n^{10})$$

# Root vs log



$$\sqrt{n} \in \omega(\lg n) \subseteq \Omega(\lg n)$$

# little- $\omega$ redux

$$f(n) \in \omega(g(n))$$

$$\forall c > 0 \exists n_0 \forall n > n_0 \quad cg(n) < f(n)$$

$$\text{equivalently } \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$$

(Assume positive functions)

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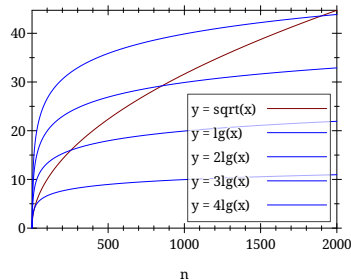
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## idea of equivalence

$$f(n) > cg(n) \Leftrightarrow \frac{f(n)}{g(n)} > c$$

(Assume positive functions)



# Limit of ratios rule

From CLRS (3.13) for  $a > 1$ :

$$\lim_{n \rightarrow \infty} \frac{a^n}{n^b} = \infty$$



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i.e.

$$(1.01)^n \in \omega(n^{10})$$

# Proving the limit of ratios rule

$$\lim_{n \rightarrow \infty} \frac{a^n}{n^b} =$$
$$=$$
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## L'Hôpital's rule

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

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$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a^n}{n^b} &= \lim_{n \rightarrow \infty} \frac{a^n \ln a}{bn^{b-1}} \\ &= \\ &= \end{aligned}$$

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## Another limit ratio rule

$$\begin{aligned}\lim_{m \rightarrow \infty} \frac{m^c}{(\lg m)^b} &= \\ &= \\ &= \infty \quad (\text{apply lrr})\end{aligned}$$

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$$\begin{aligned}\lim_{m \rightarrow \infty} \frac{m^c}{(\lg m)^b} &= \lim_{m \rightarrow \infty} \frac{(2^c)^{\lg m}}{(\lg m)^b} \\ &= \\ &= \infty \quad (\text{apply lrr})\end{aligned}$$

$$m = 2^{\lg m}$$

$$n :=$$

$$a :=$$

$$c > 0 \implies$$

$$m \rightarrow \infty \implies$$



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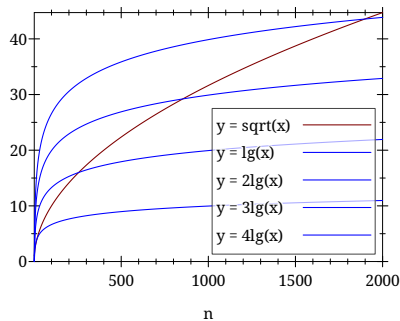
$$a := 2^c$$

$$c > 0 \implies a = 2^c > 1$$

$$m \rightarrow \infty \implies n = \lg m \rightarrow \infty$$

# Log vs Root revealed

$$\sqrt{n} \in \omega(\lg n) \subseteq \Omega(\lg n)$$



# Log vs Root revealed

$$\sqrt{n} \in \omega(\lg n) \subseteq \Omega(\lg n)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\lg n} &= \lim_{n \rightarrow \infty} \frac{n^{1/2}}{(\lg n)^1} \\ &= \infty \end{aligned} \quad (\text{LRR2})$$

