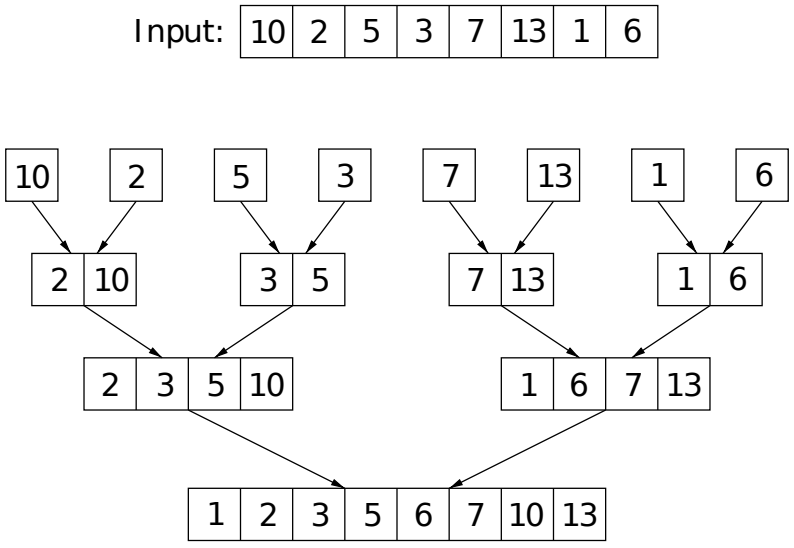
CS3383 Unit 1 Lecture 1: Divide and Conquer intro

David Bremner David Bremner

2024-01-12

Outline

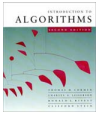
Merge Sort


Input: 10 2 5 3 7 13 1 6
10 2 5 3 7 13 1 6
2 10 3 5 7 13 1 6
2 3 5 10 1 6 7 13
1 2 3 5 6 7 10 13

Merge Sort

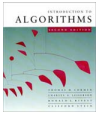
```
def merge_sort(A):
    n = len(A)
    if (n <= 1):
        return A

    split = max(n//2,1)
    left=merge_sort(A[0:split])
    right=merge_sort(A[split:])
    return merge(left,right)
```



Merging two sorted arrays

20	12
13	11
7	9
2	1



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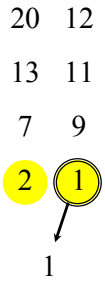

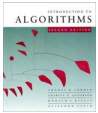

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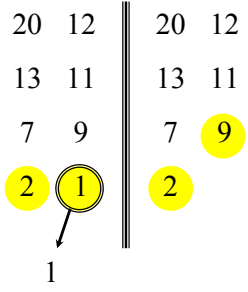

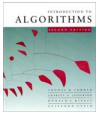

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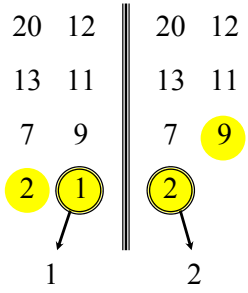

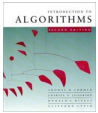

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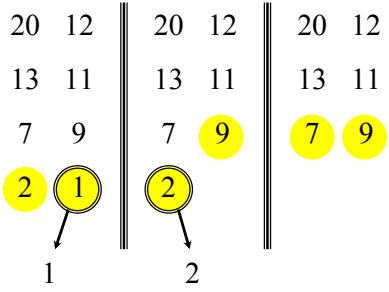

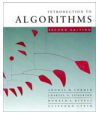

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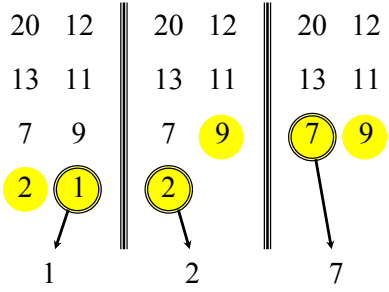

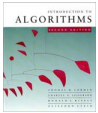

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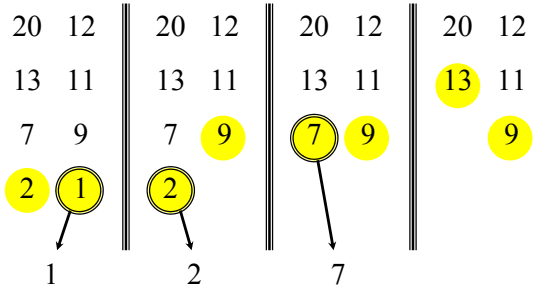

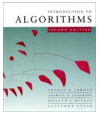

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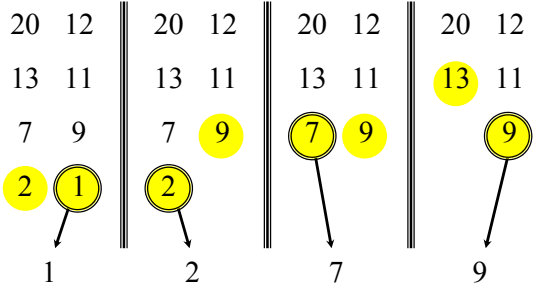

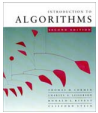

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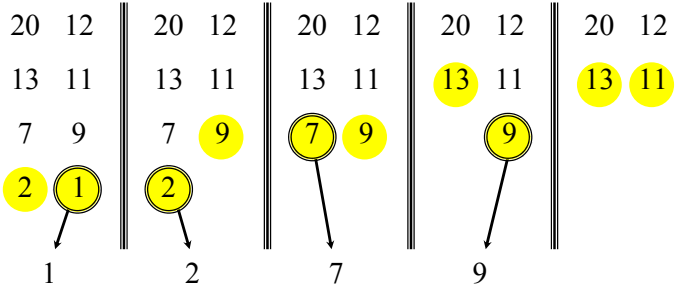

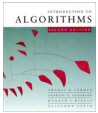

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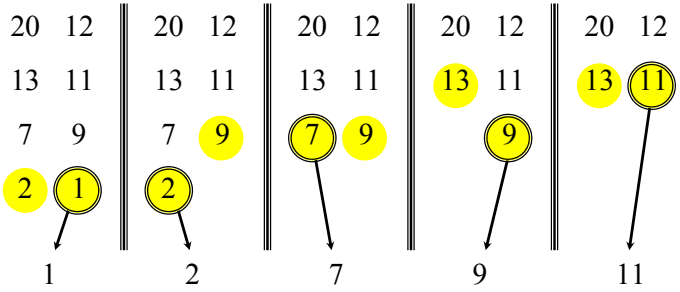

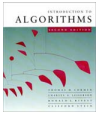

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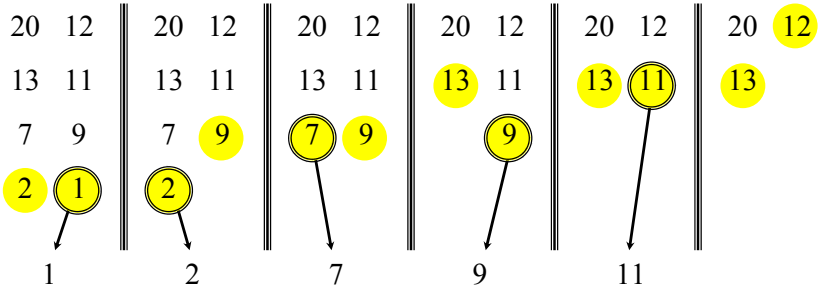

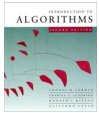

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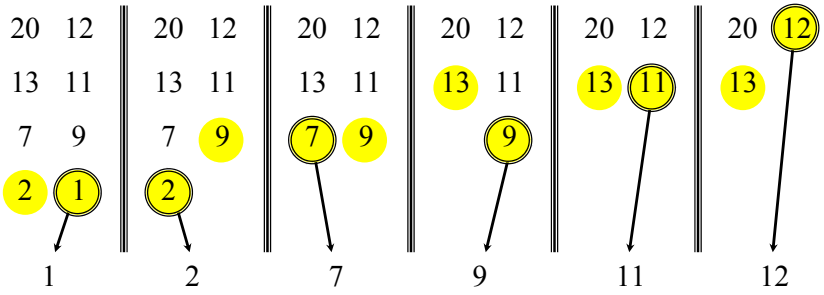

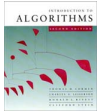

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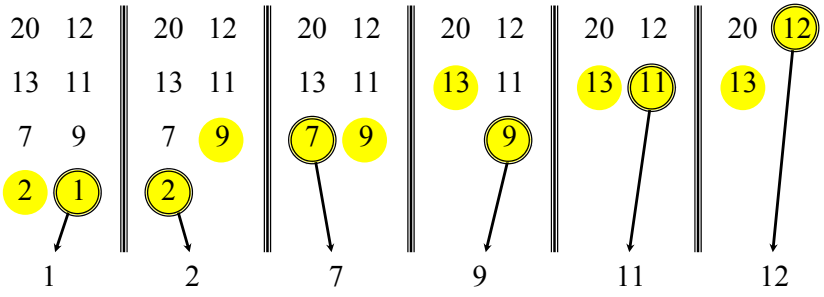
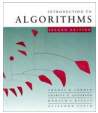


Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing Merge sort

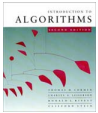
```
def merge_sort(A):
    n = len(A)                          # Θ(1)
    if (n <= 1):                        # Θ(1)
        return A                        # Θ(1)
    split = max(n//2,1)                 # Θ(1)

    left=merge_sort(A[0:split])         # T(n/2)
    right=merge_sort(A[split:])         # T(n/2)
    return merge(left,right)            # Θ(n)
```



Recurrence for merge sort

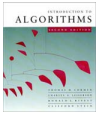
$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- We will see several ways starting with "Rec. Tree" to find a good upper bound on $T(n)$.



Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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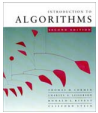
$T(n)$



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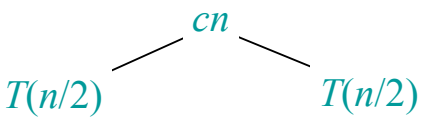

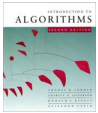

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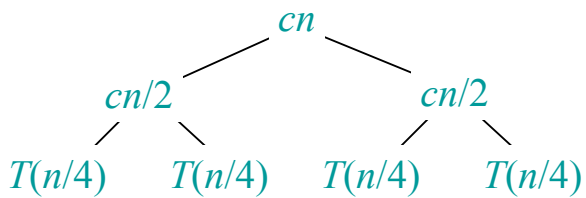

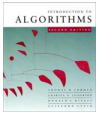

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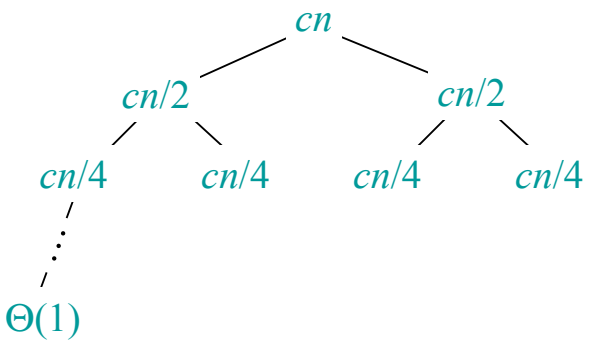

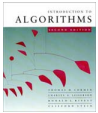

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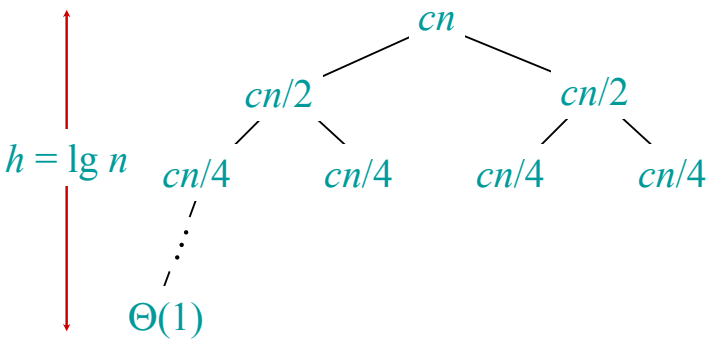

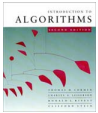

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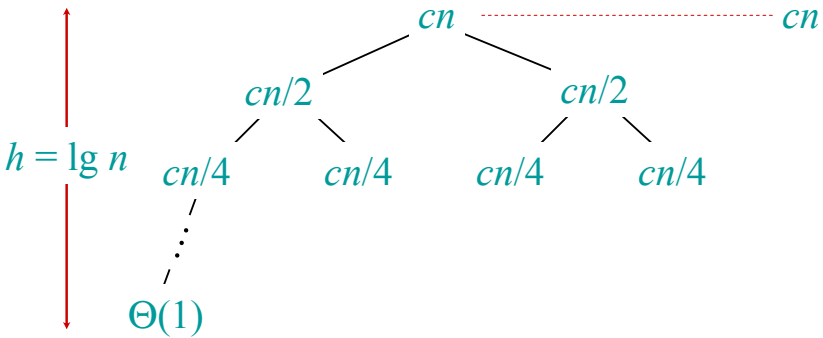

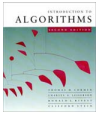

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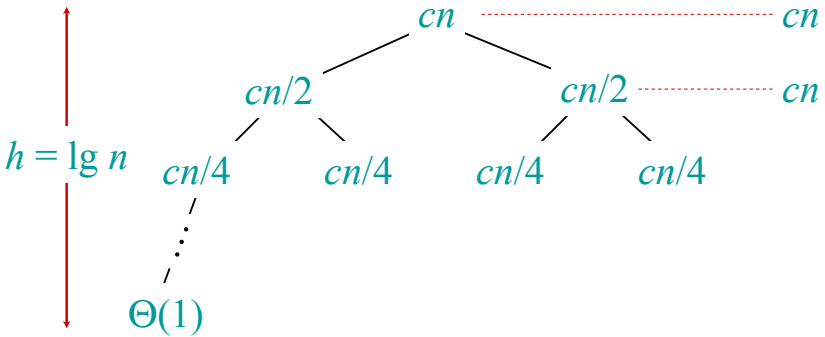

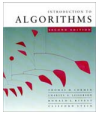

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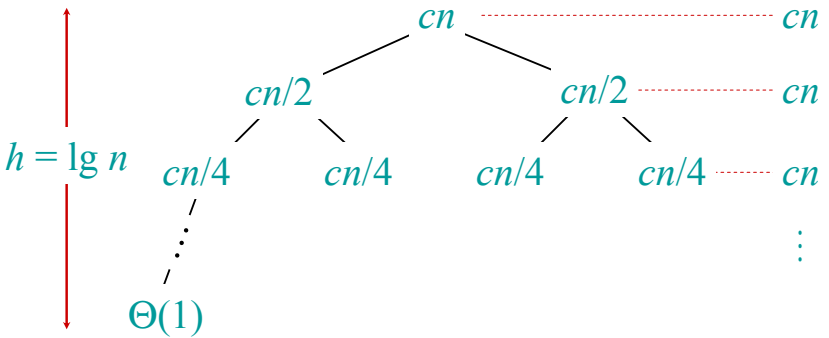

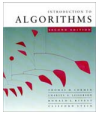

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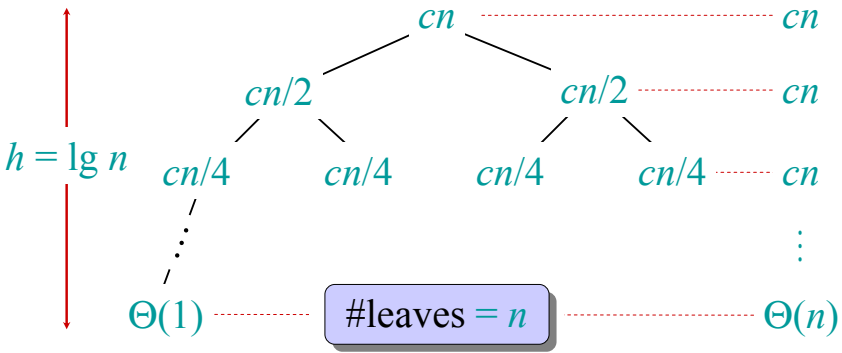

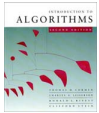

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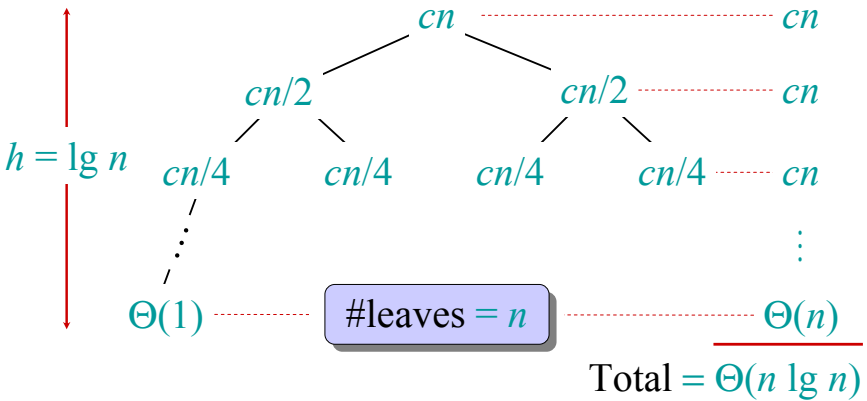
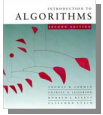
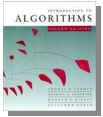




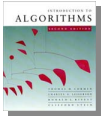
Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



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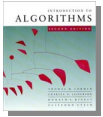

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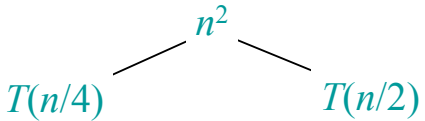

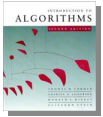

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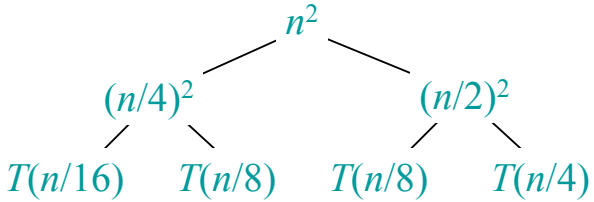

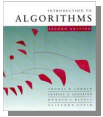

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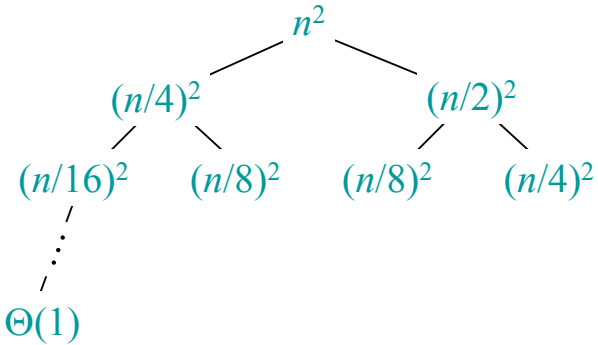

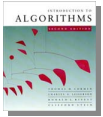

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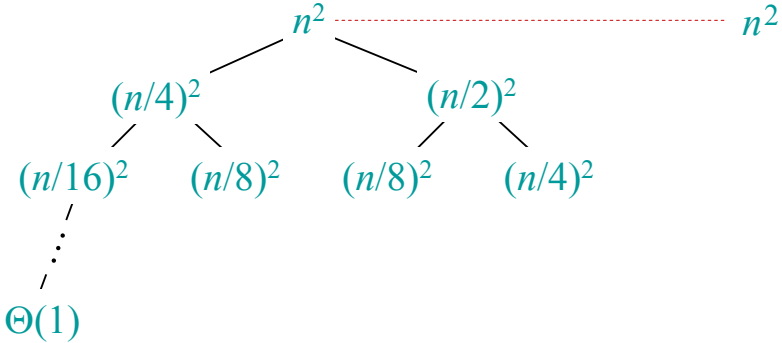

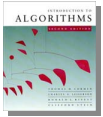

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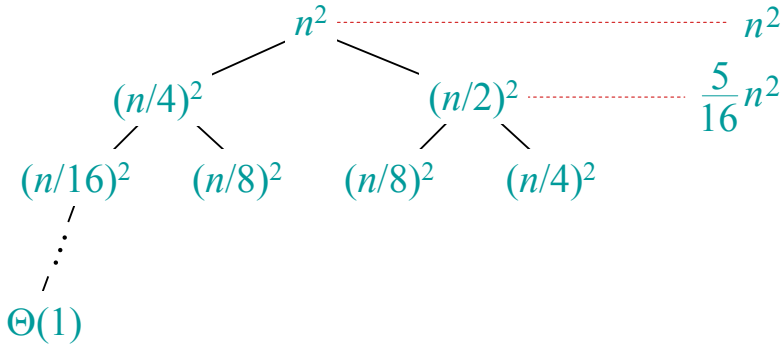

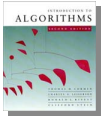

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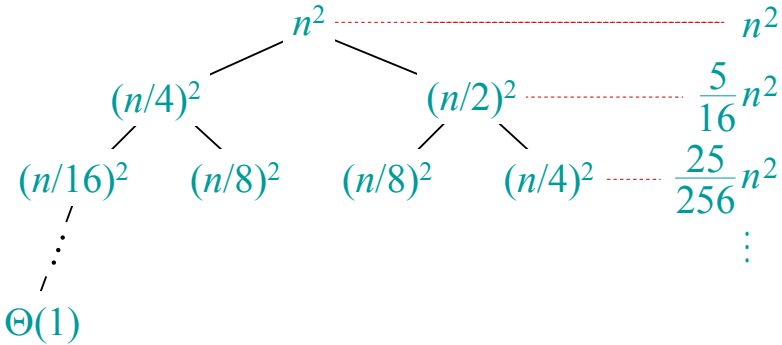

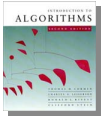

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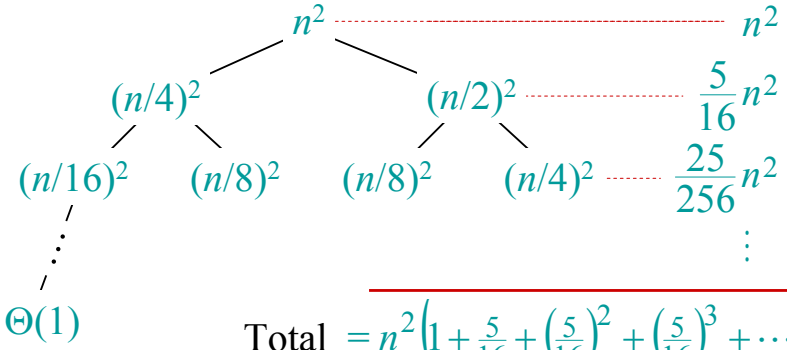
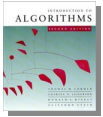


$$\text{Total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$
$$= \Theta(n^2) \quad \textit{geometric series}$$



Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

Return to last slide viewed.

Naive D&C multiplication

Solve $T(n) = 4T(\frac{n}{2}) + cn$, via recursion tree.

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$$= 2cn \cdot n - cn = 2cn^2 - cn \in \Theta(n^2)$$

Smart D&C multiplication, sloppy analysis

Solve $T(n) = 3T(\frac{n}{2}) + cn$, via recursion tree.

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$$= 3cn \cdot n^{\lg 3/2} - 2cn \in O(n^{1.6})$$

recursion tree example I/II

$$T(n) = 2T(3n/8) + n^2$$

recursion tree example I/II

$$T(n) = 2T(3n/8) + n^2$$

$$n^2, (3n/8)^2, [(3/8)^2 n]^2, [(3/8)^3 n]^2, \ldots, (3/8)^{2k} n^2$$

(call at depth k)

recursion tree example I/II

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There are 2^k nodes on level k, so work on level k is

$$2^k \cdot (3/8)^{2k} n^2 =$$

$$=$$

$$=$$

recursion tree example I/II

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$$2^k \cdot (3/8)^{2k} n^2 = 2^k \frac{3^{2k}}{2^{6k}} n^2$$

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recursion tree example I/II

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recursion tree example I/II

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$$\begin{aligned} 2^k \cdot (3/8)^{2k} n^2 &= 2^k \frac{3^{2k}}{2^{6k}} n^2 \\ &= 2^k \frac{9^k}{64^k} n^2 \\ &= \left(\frac{9}{32}\right)^k n^2 \end{aligned}$$

recursion tree example II/II

Total work

$$T(n) = n^2 \sum_{i=0}^{\log_{8/3} n} \left(\frac{9}{32}\right)^k$$

$$\leq$$

$$=$$

$$= \quad n^2$$

recursion tree example II/II

Total work

$$
\begin{aligned}
T(n) &= n^2 \sum_{i=0}^{\log_{8/3} n} \left(\frac{9}{32}\right)^k \\
&\leq n^2 \sum_{i=0}^{\infty} \left(\frac{9}{32}\right)^k \\
&= \\
&= \quad n^2
\end{aligned}
$$

recursion tree example II/II

Total work

$$
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&= n^2 \frac{1}{1 - 9/32} \\
&= \quad n^2
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recursion tree example II/II

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&= n^2 \frac{1}{1 - 9/32} \\
&= \frac{32}{23} n^2
\end{aligned}
$$