CS3383 Lecture 1.2: Substitution method

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Outline

Even More Divide and Conquer

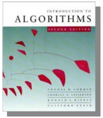
- Substitution Method for recurrences
- Substitution examples

Example recurrence

- The Master Method actually works for this, but it won't always.

$$T(n) = 4T(n/2) + n$$
$$T(1) = 1$$

- Suppose that we want to prove $T(n) \in O(n^3)$ by induction
- Guess $T(n) \leq cn^3$



Example of substitution

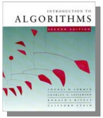
$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \leftarrow \textit{desired} - \textit{residual} \\
&\le cn^3 \leftarrow \textit{desired}
\end{aligned}
$$

whenever $(c/2)n^3 - n \ge 0$, for example, if $c \ge 2$ and $n \ge 1$.

↖ *residual*

Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.



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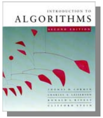
This bound is not tight!

A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.



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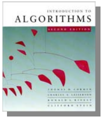
We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \le ck^2$ for $k < n$:

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\le 4c(n/2)^2 + n \\ &= cn^2 + n \\ &= O(n^2) \end{aligned}$$



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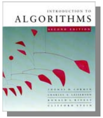
Wrong! We must prove the I.H.



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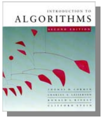
$$= cn^2 - (-n)$$ [desired – residual]

$$\le cn^2$$ for ***no*** choice of $c > 0$. Lose!



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IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.

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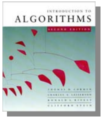
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&= 4(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1n^2 - 2c_2n + n \\
&= c_1n^2 - c_2n - (c_2n - n) \\
&\le c_1n^2 - c_2n \quad \text{if } c_2 \ge 1.
\end{aligned}
$$



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\end{aligned}
$$

Pick c_1 big enough to handle the initial conditions.

Substitution example II

$$
\begin{aligned}
T(0) &= 1 \\
T(n) &= T(n-1) + c^n && n > 0, c > 1 \\
&= T(n-2) + c^{n-1} + c^n \\
&= \sum_{i=0}^{n} c^i && \text{guess!} \\
&= \frac{c^{n+1} - 1}{c - 1} && \text{geo. series}
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$$

base T(0)

induction apply recurrence to prove GS formula

Substitution example III

$$\begin{aligned}
T(n) &= T(n/5) + T(3n/4) + cn \\
T(n) &\le dn, n \ge n_0 && \text{(Guess)} \\
&\le (1/5)dn + (3/4)dn + cn && \text{(Strong induction)} \\
&\le dn && \text{(Choice of } c\text{, } d\text{)}
\end{aligned}$$