CS3383 Lecture 1.3: Quicksort

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January 29, 2024

Outline

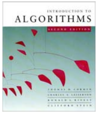
Even More Divide and Conquer

- Quicksort I: partitioning
- Quicksort II: random pivot

Quicksort demo

```
def quicksort(A,p,q):
    if p>=q: return

    r = partition(A,p,q,randrange(p,q))
    quicksort(A,p,r-1)
    quicksort(A,r+1,q)
```

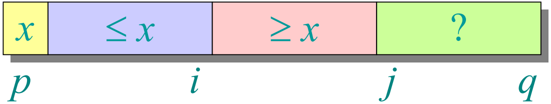


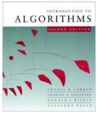
Partitioning subroutine

PARTITION(A, p, q) ▷ $A[p \,..\, q]$
  $x \leftarrow A[p]$ ▷ pivot $= A[p]$
  $i \leftarrow p$
  for $j \leftarrow p + 1$ **to** q
    do if $A[j] \leq x$
      then $i \leftarrow i + 1$
        exchange $A[i] \leftrightarrow A[j]$
  exchange $A[p] \leftrightarrow A[i]$
  return i

Running time $= O(n)$ for n elements.

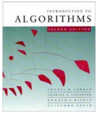
Invariant:





Example of partitioning

6	10	13	5	8	3	2	11
i	*j*						



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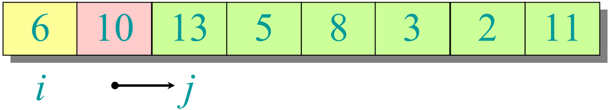

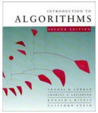

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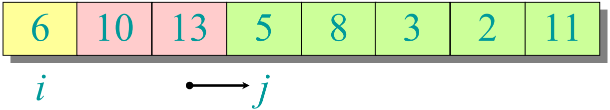

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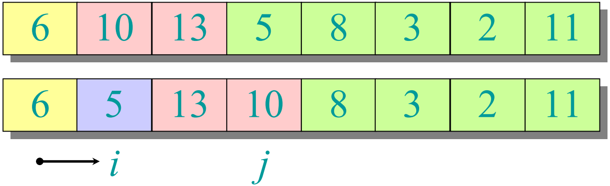
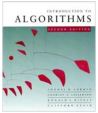


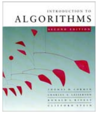


Example of partitioning

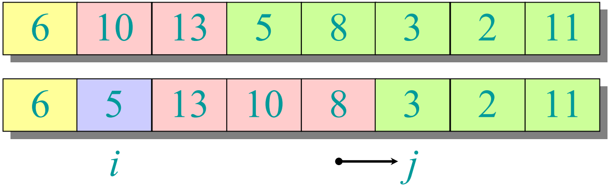
6	10	13	5	8	3	2	11

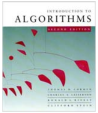
6	5	13	10	8	3	2	11

i → j

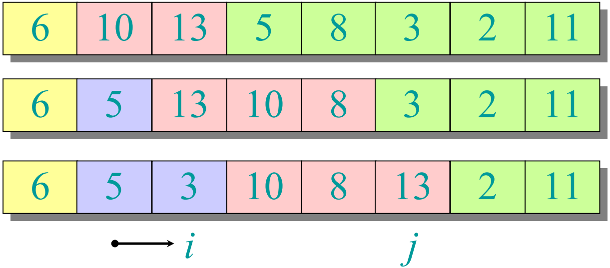


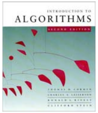
Example of partitioning





Example of partitioning





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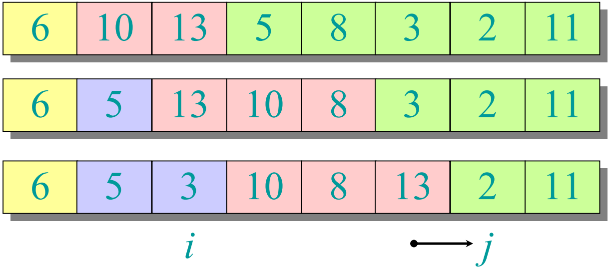


Example of partitioning

6	10	13	5	8	3	2	11

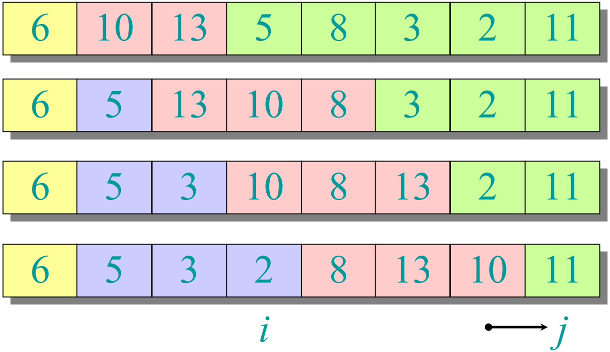
6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

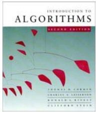
$\longrightarrow i$ $\qquad\qquad j$

Example of partitioning



Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
			i				→ j



Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11

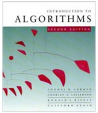
i

Random pivot

- `pivot ←A[random(1..n)]`
- indicator $X_k = 1$ if we generate a $k : n - k - 1$ split, 0 otherwise
- $E[X_k] = Pr[X_k = 1] = 1/n$, assuming distinct elements.

Random pivot demo

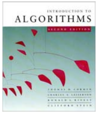
```python
def partition(A,p,q,loc):
    A[p],A[loc] = A[loc],A[p]
    i = p
    for j in range(p+1,q+1):
        if A[j] <= A[p]:
            i += 1
            A[i], A[j] = A[j], A[i]
    A[i], A[p] = A[p], A[i]
    return i
```



Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

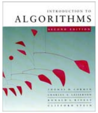
$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)$$



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

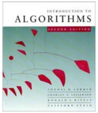


Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

Linearity of expectation.



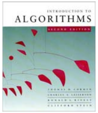
Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\bigl(T(k) + T(n-k-1) + \Theta(n)\bigr)\right]$$

$$= \sum_{k=0}^{n-1} E\bigl[X_k\bigl(T(k) + T(n-k-1) + \Theta(n)\bigr)\bigr]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of X_k from other random choices.



Calculating expectation

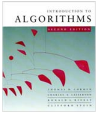
$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

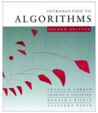
$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)$$

Summations have identical terms.



Hairy recurrence

$$E[T(n)] = \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a n \lg n$ for constant $a > 0$.

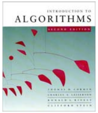
- Choose a large enough so that $a n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).

Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

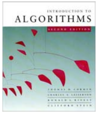


Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.



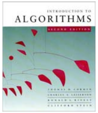
Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired*** – ***residual***.



Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

$$\le an\lg n\,,$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.