CS3383 Unit 3.2: Dynamic Programming Examples

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Outline

Longest Increasing Subsequence problem

Input Integers $a_1, a_2 \ldots a_n$

Output

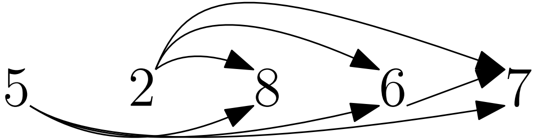
$$a_{i_1}, a_{i_2}, \ldots a_{i_k}$$

Such that

$$i_1 < i_2 \cdots < i_k$$

and

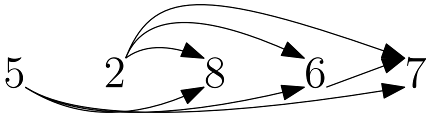
$$a_{i_1} < a_{i_2} < \cdots < a_{i_k}$$



- $(a_i, a_j) \in E$ if $i < j$ and $a_i < a_j$.
- DPV 6.2, JE 3.6

Defining subproblems

- Define $F(i)$ as the length of longest sequence starting at position i



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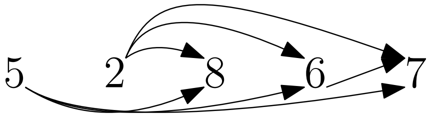

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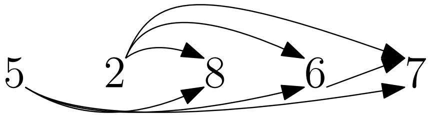

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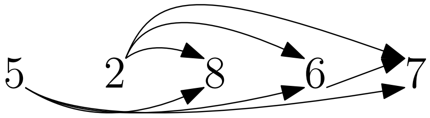
$$F(i) = 1 + \max\{F(j) \mid (i,j) \in E\}$$



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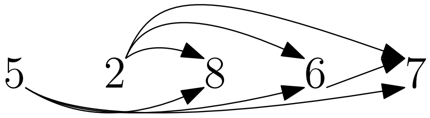
- We could solve this reasonably fast e.g. by memoization.



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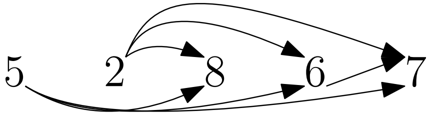
Longest path in DAG, working backwards

- Define $L[i]$ as the longest path *ending* at a_i




```
For i = 1…n:
   L[i] = 1 + max { L(j) | (j,i) in E }
```

- total cost is $O(|E|)$, *after* computing E.

Improving memory use

- We can inline the definition of E.

```
def lis(A):
  n = len(A)
  L = [1 for j in range(n)]
  for i in range(n):
    for j in range(i):
      if A[j] < A[i]:
        L[i] = max(L[i],L[j]+1)
  return max(L)
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Improving memory use

- We can inline the definition of E.
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Edit (Levenshtein) Distance

- CLRS 14-5, DPV 6.3, JE3.7
- Minimum number of insertions, deletions, substitutions to transform one string into another.

Example: timberlake $\longrightarrow$ fruitcake

- Using mostly insertions and deletions

```
i i i i   d d d d d s
_ _ _ _ T I M B E R L A K E
F R U I T _ _ _ _ _ C A K E
```

Total cost 10.

Edit (Levenshtein) Distance

- CLRS 14-5, DPV 6.3, JE3.7
- Minimum number of insertions, deletions, substitutions to transform one string into another.

Example: timberlake $\longrightarrow$ fruitcake

- Using more substitutions

```
s s s s s d s
T I M B E R L A K E
F R U I T _ C A K E
```

Total cost 7.

Alignments (gap representation)

```
1 1 1 1 0 1 1 1 1 1 1 0 0 0
_ _ _ _ T I M B E R L A K E
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```

- top line has letters from A, in order, or _
- bottom line has has letters from B or _
- cost per column is 0 or 1.

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Theorem (Optimal substructure)

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proof.

By contradiction □

Subproblems (prefixes)

- Define $E[i,j]$ as the minimum edit cost for $A[1 \dots i]$ and $B[1 \dots j]$

$$E[i,j] = \begin{cases} E[i,j-1]+1 & \text{insertion} \\ E[i-1,j]+1 & \text{deletion} \\ E[i-1,j-1]+1 & \text{substitution} \\ E[i-1,j-1] & \text{equality} \end{cases}$$

justification.

We know deleting a column removes an element from one or both strings; all edit operations cost 1. □

order of subproblems

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- dependency of subproblems is *exactly* the same as LCS, so essentially the same DP algorithm works.
- or just memoize the recursion
- what are the base cases?

Edit distance

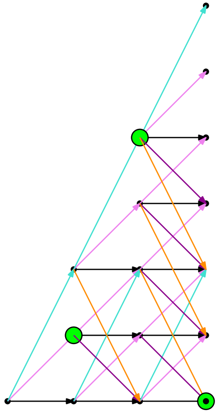
```
def dist(x,y):
  n = len(x); m=len(y)
  E = [ [max(i,j) for j in range(m+1)]
          for i in range(n+1) ]
  for i in range(1,n+1):
    for j in range(1,m+1):
      diff = int(x[i-1] != y[j-1])
      E[i][j] = min(E[i-1][j-1]+diff,
                    E[i-1][j]+1,
                    E[i][j-1]+1)
  return E
```

Tracing back the edits

```
def trace(E,x,y,i,j):
  if (i<1):
    return "i" * j;
  elif (j<1):
    return "d" * i;
  elif x[i-1] == y[j-1]:
    return trace(E,x,y,i-1,j-1)+"."
  elif E[i][j] == E[i-1][j-1] + 1:
    return trace(E,x,y,i-1,j-1)+"s"
  elif E[i][j] ==  E[i-1][j]+1:
    return trace(E,x,y,i-1,j)+ "d"
  else:
    return trace(E,x,y,i,j-1)+ "i"
```

Balloon Flight Planning

- Start at $(0, 0)$



Balloon Flight Planning

- Start at $(0, 0)$
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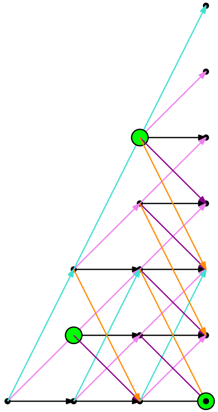

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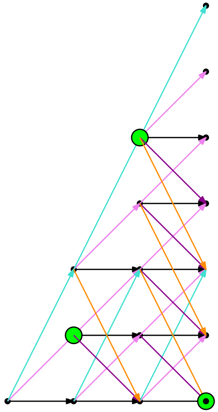

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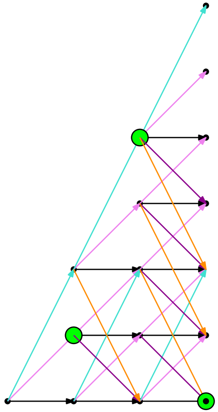

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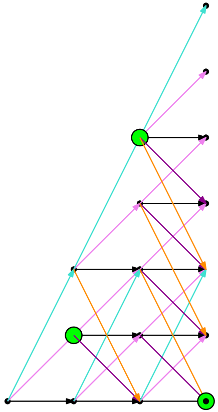
- Maximize value of collected prizes
- We can discretize/simulate the problem as a graph search



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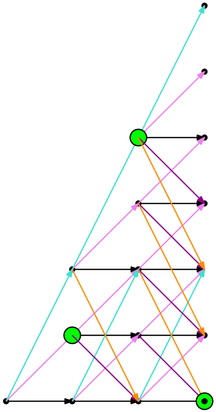

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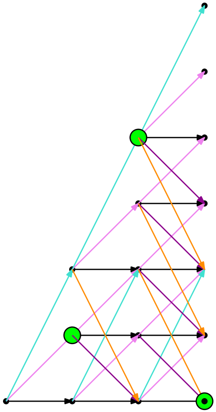

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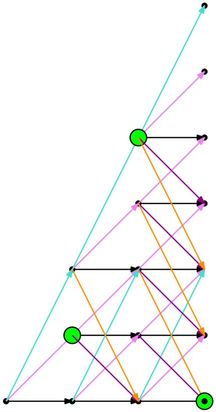

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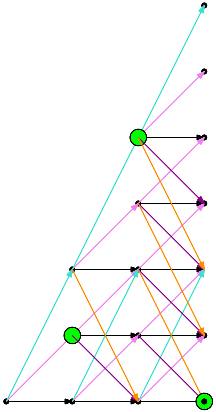

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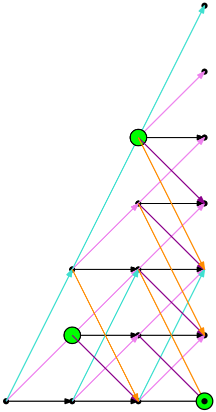
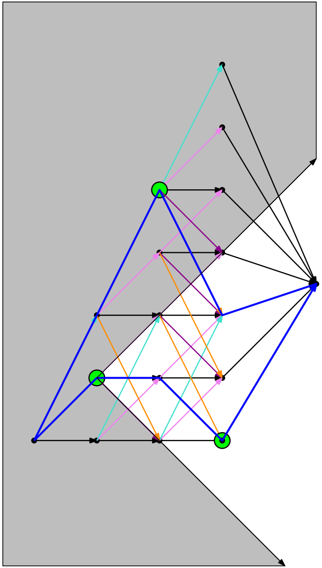
- This means we have a bad dependence on k; more about this later



Straightening paths



Lemma (Straightening Paths)

There is a feasible path from p to q iff the segment $[p, q]$ is feasible.

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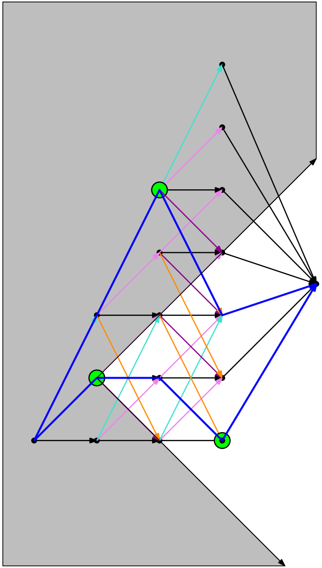

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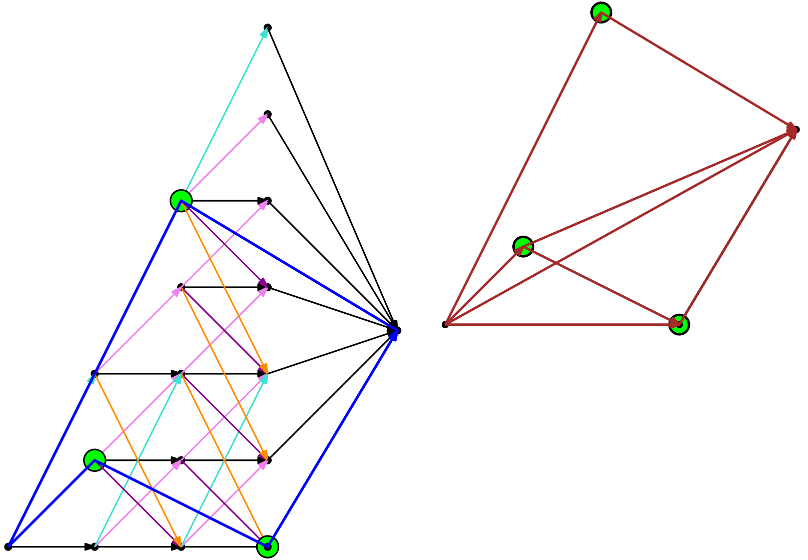
Proof sketch

The path cannot escape the cone define by the steepest possible segments.
There is always one step back towards start within cone. Apply induction.

A new graph



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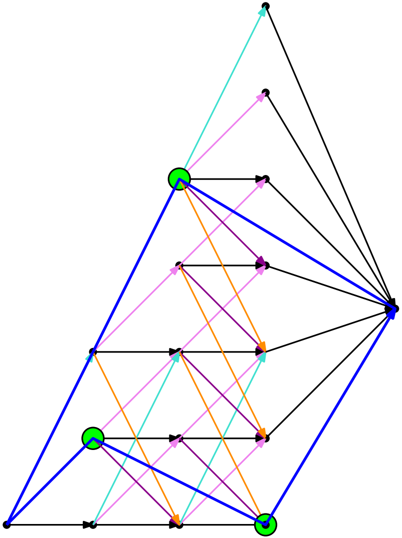

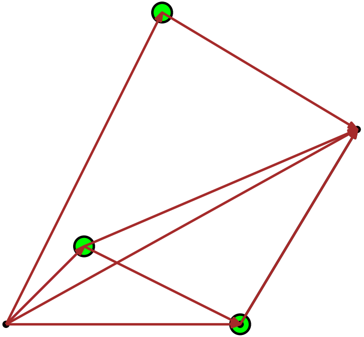

Improved graph size

The new graph is $O(p^2)$, where $p \leq n$ is the number of prizes.