THE ULTIMATE STRATEGY TO SEARCH ON *m* RAYS?

by

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TR98-118, February 1998

This research is supported by the DFG-Project
"Diskrete Probleme", No. Ot 64/8-1

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The Ultimate Strategy to Search on m Rays?[1]

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[1]This research is supported by the DFG-Project "Diskrete Probleme", No. Ot 64/8-1.

Abstract

We consider the problem of searching on m current rays for a target of unknown location. If no upper bound on the distance to the target is known in advance, then the optimal competitive ratio is $1 + 2m^m/(m-1)^{m-1}$. We show that if an upper bound of D on the distance to the target is known in advance, then the competitive ratio of any search strategy is at least $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ which is also optimal—but in a stricter sense.

We construct a search strategy that achieves this ratio. Our strategy works equally as well for the unbounded case, and produces a strategy where the point is found at a competitive distance of $1 + 2m^m/(m-1)^{m-1} - O(1/log^2 D)$, for unknown, unbounded D, that is, it is not necessary for our strategy to know an upper bound on the distance D in advance.

1 Introduction

Searching for a target is an important and well studied problem in robotics. In many realistic situations the robot does not possess complete knowledge about its environment, for instance, the robot may not have a map of its surroundings, or the location of the target may be unknown [BRS93, CL93, DHS95, DI94, IK95, Kle92, Kle94, LOS95, MI94, PY89].

Since the robot has to make decisions about the search based only on the part of its environment that it has explored before, the search of the robot can be viewed as an *on-line* problem. One way to judge the performance of an on-line search strategy is to compare the distance traveled by the robot to the length of the shortest path from its starting point s to the target t. The ratio of the distance traveled by the robot to the optimal distance from s to t over all possible locations of the target is called the *competitive ratio* of the search strategy [ST85].

We are interested in obtaining upper and lower bounds for the competitive ratio of searching on m concurrent rays. Here a point robot is imagined to stand at the origin of m rays and one of the rays contains the target t whose distance to the origin is unknown. The robot can only detect t if it stands on top of it. It can be shown that an optimal strategy visits the rays in cyclic order and increases the step length each time by a factor of $m/(m-1)$ starting with a step length of 1. The competitive ratio C_m achieved by this strategy is given by

$$1 + 2\frac{m^m}{(m-1)^{m-1}}.$$

One of the earliest references to this problem dates back to 1963 to a problem posed by Richard Bellman which assumes a probabilistic setting rather than a game theoretic one [Bel63]. Since then numerous results have been obtained [Bec64, Bec65, BN70, BW72, Gal72, Gal74, GC76] culminating in a monograph by S. Gal in 1980 [Gal80]. This monograph contains, among many other results, an optimal deterministic as well as an optimal randomized strategy to search on m rays and the corresponding lower bounds. The optimal deterministic and randomized strategies were later rediscovered [BYCR93, KRT93].

The lower bound for searching in m rays has proven to be a very useful tool for proving lower bounds for searching in a number of classes of simple polygons, such as star-shaped polygons [LO96], generalized streets [DI94, LOS96], HV-streets [DHS95], and θ-streets [DHS95, Hip94].

However, the lower bound proven for the m way ray searching problem relies on the unboundedness of the rays, that is, on the fact that the target can be placed arbitrarily far away from the starting point of the ray. But, if we consider polygons, then it is possible for the robot to obtain an upper bound D on the distance to the target. In this paper we investigate the question if the knowledge of an upper bound on the distance to the target provides an advantage for the robot. If C_m^D is the optimal competitive ratio to search on m rays where the distance to the target is at most D, then it can be expected that C_m^D approaches C_m as D goes to infinity; yet, there is only a proof for the case $m = 2$ by López-Ortiz who shows that

$$9 - O(1/\log D)$$

is a lower bound for the competitive ratio of searching on two rays [LO96]. In a similar vein Icking *et al.* investigate the maximal *reach* of a strategy to search on the line if the competitive ratio of the strategy is given [IKL97]. Here, the reach of a strategy X is the maximum distance d such that a target placed somewhere in the interval $[1, d]$ on the left or right hand side of the origin is detected by a robot using d. Given the competitive ratio C an expression for the reach is derived and it is shown that the reach is monotone [IKL97].

In this paper we prove that

$$1 + 2\frac{m^m}{(m-1)^{m-1}} - O(\frac{1}{\log^2 D})$$

is a lower bound for the competitive ratio of searching on m rays which also improves López-Ortiz' bound for $m = 2$. Moreover, we present a strategy that achieves a competitive ratio of

$$1 + 2\frac{m^m}{(m-1)^{m-1}} - O(\frac{1}{\log^2 D})$$

if the target is discovered at distance D. Astonishingly, our strategy achieves this competitive ratio without knowing an upper bound on the distance to the target in advance. The two results imply that knowing an upper bound on the distance in advance does not improve the competitive ratio significantly. Note that all previously known strategies have a competitive ratio of

$$1 + 2\frac{m^m}{(m-1)^{m-1}} - O(\frac{1}{D})$$

if the target is detected at distance D.

The paper is organized as follows. In the next section we introduce some definitions and give some introductory examples in order to motivate our approach. In Section 3 we introduce the problem of searching on m rays if a bound on the maximum distance to the target is given. In Section 4 we first consider searching on two rays to introduce our approach. In Section 5 we generalize the ideas of the case of searching on two rays to m rays and also prove a lower bound. Finally, in Section 6 we present a strategy whose competitive ratio converges asymptotically as fast to $1 + 2m^m/(m-1)^{m-1}$ as the lower bound we have shown.

2 Definitions

Let X be a strategy to search on m rays. We model X as a sequence of positive real numbers, that is, $X = (x_0, x_1, x_2, \dots)$ with $x_k > 0$, for all $0 \leq k < \infty$. We illustrate this for the case of a point robot searching on the real line, that is, $m = 2$.

In the beginning the position of the robot is a point s on the real line; it has to find a target t that is located somewhere to its left or right. It can only detect t if it stands on top of it. The robot starts at the origin s and travels to one side, say to the left.

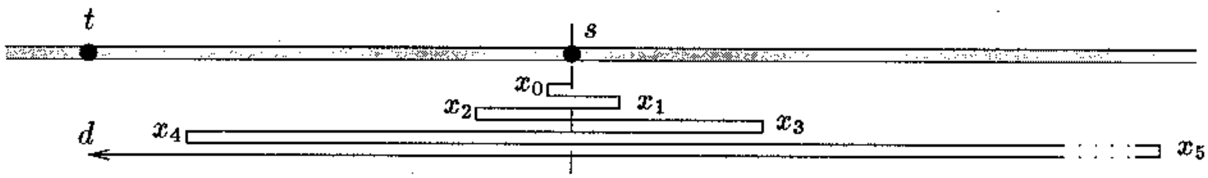

Figure 1: Searching on the real line.

At some point, say at a distance of x_0 to s, it decides that it has traveled far enough to the left and turns around. Since the target is not between its turn point and s, the only reasonable strategy for the robot is to return to the origin and explore some part of the line to the right of s. After having traveled a distance of x_1 to the right, the robot turns around again and returns to s to explore the left side again and so on. For illustration see Figure 1. Obviously, the values x_i which denote the distance that the robot travels to the left or to the right of s—depending on whether i is even or odd—suffice to characterize a search strategy completely.

The Competitive Ratio

Assume that the target is discovered in Step $k+2$, say to the left of the origin. Clearly, the ray to the left of the origin was visited the last time before Step $k+2$ in Step k. Hence, the distance d to the target is greater than x_k. The distance traveled by the robot to discover t is $d + 2\sum_{i=0}^{k+1} x_i$. Hence, the competitive ratio of Step k is

$$\frac{d + 2\sum_{i=0}^{k+1} x_i}{d} = 1 + 2\frac{\sum_{i=0}^{k+1} x_i}{d}$$

with $d > x_k$. Since d can be placed arbitrarily close to x_k by an adversary, the highest lower bound on the competitive ratio of Step k is given by the expression

$$\sup_{d > x_k} 1 + 2\frac{\sum_{i=0}^{k+1} x_i}{d} = 1 + 2\frac{\sum_{i=0}^{k+1} x_i}{x_k}.$$

Note that the above expression only depends on elements of X.

The First Step

If we consider searching on the line, then the first step is a special case that we have not considered yet. If no information about the target is available, then one false move in the beginning may lead to an arbitrarily large competitive ratio since no matter how small x_0 is chosen, we can always place the target at a distance of εx_0 to s on the opposite side, for some $\varepsilon > 0$. The competitive ratio $1 + 2x_0/\varepsilon x_0 = 1 + 2/\varepsilon$ can become arbitrarily large in this way. In order to avoid this problem we assume that a *lower bound* d_{min} for the distance to the target t is known in advance. In applications such a lower bound is

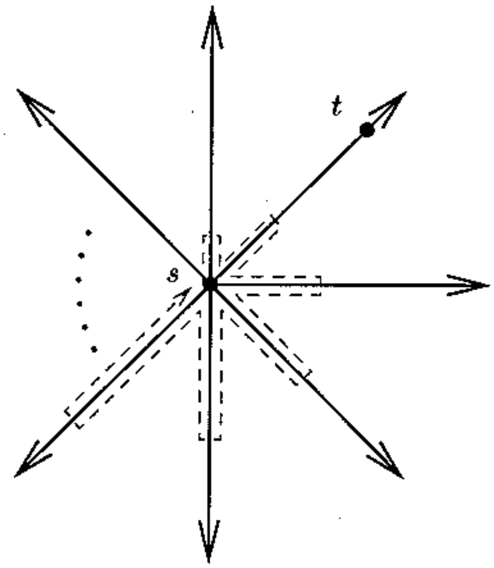

Figure 2: Searching on m rays.

usually known or can easily be computed. Hence, the competitive ratio C_X of strategy X for searching on the real line is given by

$$C_X = \max\left\{1 + \frac{2x_0}{d_{min}}, \sup_{k\geq 0} 1 + 2\frac{\sum_{i=0}^{k+1} x_i}{x_k}\right\}. \quad (0.1)$$

We can assume in the following that $d_{min} = 1$ since if we multiply both the sequence X and the initial lower bound d_{min} by a positive number, then the competitive ratio of X does not change.

3 Searching on m Rays

Searching on the real line can be viewed as searching on the rays to the left and right of s. Hence, it is natural to allow more than two rays to meet at s. So consider m concurrent rays meeting at s, one of which contains the target t (see Figure 2). It can be shown that the strategy that increases the step length each time a new ray is visited by a factor of $m/(m-1)$ is optimal [Gal80, BYCR93]. Its competitive ratio C_m is given by

$$C_m = 1 + 2\frac{m^m}{(m-1)^{m-1}}.$$

We are interested in the case that an upper bound D on the maximum distance of the target to the origin is known. The target may be placed on any of the m rays somewhere in the interval $[1, D]$ where we again assume that the lower bound on the distance to the target is one. We now model a strategy X as a finite sequence of positive numbers, that is, $X = (x_0, \ldots, x_n)$, for some $n \geq 0$. We are interested in a lower bound on the competitive ratio a strategy searching m rays. We denote the competitive ratio of searching on m-rays for a target that is placed at a distance of at most D from the origin by C_m^D.

3.1 Periodicity

In order to prove lower bounds on the competitive ratio, we investigate properties of optimal strategies, that is, strategies with minimal competitive ratio. If we denote the ray that the robot visits in Step k by r_k, then a strategy is *periodic* if $r_{k+m} = r_k$, for all $0 \le k \le n-m$. In the following we show that there is an optimal strategy that is periodic. In order to do this we first show that there is an optimal strategy that is *monotone*. A strategy is *monotone* if $x_{i+1} \ge x_i$, for all $0 \le i \le n-1$.

Lemma 3.1 *There is an optimal strategy that is monotone up to the last step.*

Proof: The proof is analogous to the proof for the unbounded case $D = \infty$ (see [Gal80]). Let $X = (x_i)$ be a strategy to search m rays and r_i the ray that is explored by X in the ith step. We define $J_i = \min\{j > i \mid r_j = r_i\}$. If there is no $j > i$ with $r_j = r_i$, then we define $J_i = i$. We represent X by the sequence of pairs (x_i, J_i). If J_i does not equal i, then the competitive ratio in Step i of strategy X is given by $1 + 2F_i(X)$ where

$$F_i(X) = \frac{\sum_{j=0}^{J_i-1} x_j}{x_i},$$

which can be seen as in the case of searching on two rays. If J_i equals i, that is, $x_i = D$ and Step i is the last step on ray r_i, then the competitive ratio in Step i of strategy X is bounded by

$$\frac{2\sum_{j=0}^{i-1} x_j + d}{d} \le 1 + 2\frac{\sum_{j=0}^{i-1} x_j}{x_{J_i^{-1}}} = 1 + 2F_{J_i^{-1}}(X)$$

where J_i^{-1} is the index of the last visit of ray r_i before i, that is, $J_{J_i^{-1}} = i$ and $J_{J_i}^{-1} = i$; $d > x_{J_i^{-1}}$ is the distance from the origin to the target.

Assume that there is a Step k, $0 \le k \le n-1$ such that $x_{k+1} < x_k$. Let X' be the search strategy which is equal to X except that for all steps $i \ge k$ the role of r_k and r_{k+1} is exchanged as are x_k and x_{k+1}. This can be achieved by setting $(x'_k, J'_k) = (x_{k+1}, J_{k+1})$ and $(x'_{k+1}, J'_{k+1}) = (x_k, J_k)$. For all other Steps i, $(x'_i, J'_i) = (x_i, J_i)$. If $x'_{k+1} = D$, then we set $J'_{k+1} = k+1$ (and not equal to k as is implied by the rule above). $x'_k = x_{k+1} = D$ is not possible since $x_{k+1} < x_k \le D$. We want to show that $\max_{0 \le i \le n} F_i(X') \le \max_{0 \le i \le n} F_i(X)$. $F_i(X)$ and $F_i(X')$ differ at most for the indices J_k^{-1}, J_{k+1}^{-1}, k, $k+1$.

First we assume that Step k is not the last step on ray r_k. (Note that Step $k+1$ is not the last step on ray r_{k+1} as $x_{k+1} < x_k \le D$.) It is easy to see that in this case

$$F_k(X) = \frac{\sum_{i=0}^{J_k-1} x_i}{x_k} = \frac{\sum_{i=0}^{J'_{k+1}-1} x'_i}{x'_{k+1}} = F_{k+1}(X') \qquad \text{and}$$

$$F_{k+1}(X) = \frac{\sum_{i=0}^{J_{k+1}-1} x_i}{x_{k+1}} = \frac{\sum_{i=0}^{J'_k-1} x'_i}{x'_k} = F_k(X').$$

Here the equalities follow from the fact that $J'_{k+1} = J_k \geq k+2$ and $J'_k = J_{k+1} \geq k+2$, that is, the exchange of x_k and x_{k+1} does not play a role in the summation. Next we consider Steps J^{-1}_{k+1} and J^{-1}_k. Since $J_{J_k^{-1}} - 1 = k-1$ and $J_k^{-1} = J_k^{-1\prime}$, $F_{J_k^{-1}}(X) = F_{J_k^{-1\prime}}(X')$. This leaves us with Step J^{-1}_{k+1}. We have

$$\begin{aligned} F_{J^{-1}_{k+1}}(X) &= \frac{\sum_{i=0}^{k} x_i}{x_{J^{-1}_{k+1}}} \geq \frac{\sum_{i=0}^{k} x_i - x_k + x_{k+1}}{x_{J^{-1}_{k+1}}} \\ &= \frac{\sum_{i=0}^{k-1} x'_i + x'_k}{x'_{J^{-1}_{k+1}{}'}} = F_{J^{-1}_{k+1}{}'}(X'). \end{aligned}$$

Now assume that Step k is the last step on ray r_k and $D = x_k > x_{k+1}$. Then, $F_{k+1}(X') \leq F_{J^{-1}_{k+1}{}'}(X')$. As above we obtain $F_k(X') = F_{k+1}(X)$, $F_{J_k^{-1}}(X') = F_{J_k^{-1}}(X)$ and $F_{J^{-1}_{k+1}{}'}(X') \leq F_{J^{-1}_{k+1}}(X)$. Hence, the competitive ratio of Strategy X' is no more than the competitive ratio of strategy X.

By performing bubble-sort on strategy X we see that there is a monotone strategy that has a competitive ratio no more than X. If we choose X to be an optimal strategy, then this implies the claim. □

By Lemma 3.1 it suffices to consider monotone strategies in the following. Note that if X is monotone, then the last m steps of X all have length D, that is, there is an optimal strategy with $x_{n-m+1} = \cdots x_n = D$.

Lemma 3.2 *There is an optimal strategy that is periodic.*

Proof: Let X be an optimal strategy that is monotone which exists by Lemma 3.1. We follow the proof idea of Yin [Yin94]. Let X^* consist of the same sequence of numbers except that X^* is now considered a periodic strategy. We consider the competitive ratios C_k of X and C^*_k of X^* in Step k. It suffices to show that, for every $0 \leq k \leq n-m$, there is a $0 \leq j \leq n-m$ with $C^*_k \leq C_j$. We do not need to consider the indices $n-m+1 \leq k \leq n$ since $x_k = D$, if $n-m+1 \leq k \leq n$, and $C^*_k \leq C^*_{k-m}$. So consider

$$C^*_k = 1 + 2\frac{\sum_{i=0}^{k+m-1} x_i}{x_k},$$

for some $0 \leq k \leq n-m$. For each ray r_j, $1 \leq j \leq m$, let k_j be the first time X explores ray r_j after Step k. Since $x_j < D$, for all $0 \leq j \leq n-m$, k_j exists, for all $1 \leq j \leq n-m$. Note that there is one ray r_l such that $k_l \geq k+m$. If r_l is explored before Step k, then let $j_l \leq k$ be the index of the last exploration; otherwise let $j_l = -1$ and $x_{j_l} = 1$. In both cases $x_{j_l} \leq x_k$ since X is monotone and

$$C^*_k = 1 + 2\frac{\sum_{i=0}^{k+m-1} x_i}{x_k} \leq 1 + 2\frac{\sum_{i=0}^{k_l-1} x_i}{x_{j_l}} = C_{j_l},$$

which implies that the competitive ratio of X is at least as large as the competitive ratio of X^*. □

3.2 A Recurrence Equation

In the following we assume that X is an optimal periodic strategy. The competitive ratio of X in Step k is again given by $1 + 2F_k(X)$ where

$$F_k(X) = \frac{\sum_{i=0}^{k+m-1} x_i}{x_k},$$

for $k = 0, \ldots, n - m + 1$. Let $c_X = \max_{0 \le i \le n-m+1} F_i(X)$.

Lemma 3.3 ([KPY96]) *If X is an optimal strategy, then $F_k(X) = c_X$, for all $0 \le k \le n - m + 1$.*

Proof: The proof is by contradiction. It is based on the observation that F_k is the only function which is decreasing in x_k and all other functions F_i with $i > k$ are increasing in x_k [KPY96]. So if there is an index k with $F_k(X) < c_X$, then there is an $\varepsilon > 0$ such that if x_k is decreased by ε, then $F_k(X') = c_X$ if X' is the sequence where x_k is replaced by $x_k - \varepsilon$. The decrease in x_k also implies that $F_i(X') < c_X$, for all $k < i \neq k \le n - m + 1$.

Assume there is no optimal sequence X with $F_k(X) = c_X$, for all $0 \le k \le n - m + 1$. Then, there is a maximal l and a sequence X such that $F_k(X) = c_X$, for all $0 \le k \le l < n - m + 1$. In particular, $F_{l+1}(X) < c_X$. If we apply the above arguement, then we can construct a sequence X' from X with $F_k(X') = c_{X'}$, for all $0 \le k \le l + 1$—a contradiction to the maximality of l. □

Note that if X is an optimal strategy, then $1 + 2c_X = C_m^D$. Lemma 3.3 implies that there is a recurrence equation for X.

Corollary 3.4 *If X is an optimal strategy, then*

$$x_{k+m-1} - c_m^D x_k + c_m^D x_{k-1} = 0, \tag{0.2}$$

for all $0 \le k \le n - m$, where $c_m^D = (C_m^D - 1)/2$ and $x_{-1} = 1$.

Proof: Let X be an optimal strategy. By Lemma 3.3 we have

$$\frac{\sum_{i=0}^{k+m-1} x_i}{x_k} = c_m^D \quad \Rightarrow \quad \sum_{i=0}^{k+m-1} x_i = c_m^D x_k, \tag{0.3}$$

for $1 \le k \le n - m$. The same equation also holds for $k - 1$. Hence,

$$\sum_{i=0}^{k+m-1} x_i = c_m^D x_k \quad \text{and} \quad \sum_{i=0}^{k+m-2} x_i = c_m^D x_{k-1}.$$

By subtracting the second equation from the first we obtain the following recurrence equation

$$x_{k+m-1} - c_m^D x_k + c_m^D x_{k-1} = 0,$$

for all $1 \leq k \leq n-m$, as claimed.

Note that the robot visits the ray r_1 in Step $k-m+1$; hence, $x_{k-m+1} = D$ and F_{k-m+1} does not exist.

The first time ray m is visited, the competitive ratio is given by

$$1 + 2\frac{\sum_{i=0}^{m-1} x_i}{d_{min}} = 1 + 2c_m^D$$

where $d_{min} = 1$ is the lower bound on the distance to the target. If we set $x_{-1} = 1$, then the above equation can be rewritten as

$$\sum_{i=0}^{m-1} x_i = c_m^D x_{-1}.$$

If we substract this equation from the equation for $k = 0$, the claim follows. □

It is interesting to note that if we define $s_k = \sum_{i=0}^{k} x_i$, then Equation 0.3 can be written as

$$\frac{s_{k+m-1}}{s_k - s_{k-1}} = c_m^D \qquad \Rightarrow \qquad s_{k+m-1} - c_m^D s_k + c_m^D s_{k-1} = 0,$$

that is, the values s_i satisfy the same recurrence as the values x_i. However, we only deal with Equation 0.2 in the following since we do not know any boundary values for s_i.

Equation 0.2 defines the sequence $X = (x[-1], x_0, x_1, \ldots, x_n)$ if we are given any starting values $x_0, \ldots, x_{m-1}$. Unfortunately, we do not know the values of $x_0, \ldots, x_{m-1}$; however, we know the values of $x_{n-m+1}, \ldots, x_n$ since $x_i = D$ in the last m steps. However, only $m-1$ of these are relevant since x_n does not appear in Equations defined by (0.2). The m-th boundary value of Equation 0.2 is given by $x_{-1} = 1$.

We now transform the problem such that we obtain a recurrence equation for which m consecutive starting values are given and which still yields a lower bound for c_D^c. Consider the element x_{n-m}. We have

$$x_{n-m} = \frac{\sum_{i=0}^{n-1} x_i}{c_m^D} \qquad \text{or} \qquad (1 - 1/c_m^D)x_{n-m} \geq \frac{\sum_{i=n-m+1}^{n-1} x_i}{c_m^D} \geq \frac{m-1}{c_m^D} D$$

or

$$x_{n-m} \geq \frac{m-1}{c_m^D - 1} D \geq \frac{m/2}{em} D \geq \frac{D}{2e}.$$

So we have as initial conditions

$$\begin{aligned} x_{n-m} &\geq D/2e \\ x_{n-m+1} &= D \\ &\vdots \\ x_{n-1} &= D. \end{aligned}$$

Now let $X = (x_0, x_1, ..., x_n)$ be an optimal strategy that satisfies the recurrence equation and the above initial conditions. If we cut off the last m values to

$$\begin{aligned} x_{n-m} &= D/2e \\ x_{n-m+1} &= D/2e \\ &\vdots \\ x_{n-1} &= D/2e, \end{aligned}$$

then the new strategy does not fulfill Equation 0.2 anymore but only

$$x_{k+m-1} - c(x_k - x_{k-1}) \leq 0,$$

for all $1 \leq k \leq n-m$, where $c = c_m^D$. As in Lemma 3.3 we can now argue that there is an optimal strategy (one with minimal c) such that

$$x_{k+m-1} - c(x_k - x_{k-1}) = 0,$$

for all k, as x_k in only negative in the equation for x_{k+m-1}. Since we are only interested in the asymptotic behaviour of $m^m/(m-1)^{m-1} - c_m^D$ we neglect the division by $2e$ in the following and assume that we are given

$$\begin{aligned} x_{n-m} &= D \\ x_{n-m+1} &= D \\ &\vdots \\ x_{n-1} &= D \end{aligned}$$

as initial values for Equation 0.2. As it turns out this does not influence the asymptotic behaviour of $m^m/(m-1)^{m-1} - c_m^D$. So we are now given the *last* m values of Equation 0.2.

In order to make use of this information we consider the sequence Y of the values of X in reverse order, that is, $y_i = x_{n-i-1}$, for $i = -1, \ldots, n-1$. For simplicity we write c instead of c_m^D in the following. The values y_i satisfy the following recurrence

$$y_k - cy_{k+m-1} + cy_{k+m} = 0 \qquad \text{or} \qquad y_{k+m} - y_{k+m-1} + \frac{1}{c} y_k = 0, \tag{0.4}$$

for $0 \leq k \leq n-m$ with starting values $y_0 = \cdots = y_{m-1} = D$. The initial steps again have to be considered separately. The competitive ratio the first time the ray r_m is visited is bounded by $1 + 2\sum_{i=0}^{m-2} x_i$. Therefore, $1 + 2\sum_{i=0}^{m-2} x_i \leq 1 + 2c$ since otherwise X does not have a competitive ratio of $1 + 2c$. Hence, the value of $\sum_{i=n-m+2}^{n} y_i$ also has to be at most c. Moreover, all the values $y_0, \ldots, y_n$ have to be positive.

We assume in the following that Equation 0.4 defines an infinite sequence Y some of whose elements may be negative.

In order to prove a lower bound on the competitive ratio $1 + 2c$ we show the following theorem.

Theorem 3.5 *If $c < m^m/(m-1)^{m-1} - O(1/\log^2 D)$, then there is no sequence Y and no $n \geq 0$ such that Y satisfies Equation 0.4, $\sum_{i=n-m+2}^{n} y_i \leq c$, $y_0 = y_1 = \cdots = y_{m-1} = D$, and $y_m, \ldots, y_n \geq 0$.*

By the construction of Y we also obtain that there is no strategy X with a competitive ratio of $1 + 2c$ to search on m rays in the interval $[1, D]$.

Lemma 3.6 *If there is no sequence Y and no $n \geq 0$ such that Y satisfies Equation 0.4, $\sum_{i=n-m+2}^{n} y_i \leq c$ and $y_0 = y_1 = \cdots = y_{m-1} = D$, and $y_m, \ldots, y_n \geq 0$, then there is no strategy X with a competitive ratio of $1 + 2c$ that searches on m rays for a target of distance at most D to the origin.*

Proof: The proof is by contradiction. Assume there is a strategy X with a competitive ratio of $1+2c$ that searches on m rays for a target of distance at most D to the origin. By Lemma 3.2 and the above considerations we can assume that X is periodic and satisfies Equation 0.2. Obviously, we can cut off the last $m + 1$ steps of X to $D/2e$. Let X' be the new sequence with $x'_{-1} = 1$. Define the sequence Y by $y_i = x'_{n-i-1}$, for $0 \leq i \leq n$, where n is the length of X. The values of Y satisfy Equation 0.4 and $\sum_{i=n-m+2}^{n} y_i \leq c$ in contradiction to the assumption that no such Y exists. □

3.3 The Characteristic Equation

We only consider the sequence Y in the following. Equation 0.4 has the characteristic equation

$$\lambda^m - \lambda^{m-1} + \frac{1}{c} = 0 \qquad \text{or} \qquad c = \frac{1}{\lambda^{m-1}(1-\lambda)}. \tag{0.5}$$

We first note that since $\lambda^{m-1}(1-\lambda) < 0$, for $\lambda > 1$, there is no positive real root larger than one. On the other hand, if we set $\mu = 1/\lambda$, then $c = \mu^m/(\mu - 1)$ and if there is a positive real root λ of Equation 0.5 with $\lambda < 1$, then $c \geq \inf_{\mu > 1} \mu^m/(\mu - 1) = m^m/(m-1)^{m-1}$ and we are done. Hence, we can assume in the following that there is no positive real root of Equation 0.5.

So we investigate the complex and negative roots of Equation 0.5 in more detail.

4 Solving the Recurrence Equation for $m = 2$

In order to illustrate our approach we present the case $m = 2$ in greater detail. We assume that c is less than 4 in the following.

4.1 An Explicit Solution

For $m = 2$ Equation 0.5 reduces to

$$\lambda^2 - \lambda + 1/c = 0 \tag{0.6}$$

with the solutions

$$\lambda = \frac{1}{2}\left(1 + i\sqrt{\frac{4-c}{c}}\right) \quad \text{and} \quad \overline{\lambda} = \frac{1}{2}\left(1 - i\sqrt{\frac{4-c}{c}}\right).$$

Here, $\overline{\lambda}$ denotes the conjugate of λ. Hence, the solution to Equation 0.4 in the case $m = 2$ is given by

$$y_k = a\lambda^k + \overline{a}\overline{\lambda}^k = 2Re(a\lambda^k)$$

where Re denotes the real part of a complex number. a and $\overline{a}$ are the solutions of the equation system

$$\begin{array}{rcl} a + \overline{a} &=& y_0 = D \\ a\lambda + \overline{a}\overline{\lambda} &=& y_1 = D. \end{array}$$

We obtain as solutions for a and $\overline{a}$

$$a = \frac{D}{2}\left(1 - i\sqrt{\frac{c}{4-c}}\right) \quad \text{and} \quad \overline{a} = \frac{D}{2}\left(1 + i\sqrt{\frac{c}{4-c}}\right)$$

4.2 Polar Coordinates

If we consider the polar-coordinates of λ and $\overline{\lambda}$, that is, $\lambda = \rho e^{i\varphi}$ and $\overline{\lambda} = \rho e^{i(-\varphi)}$, then $\rho = \sqrt{1/c}$ and $\varphi = \arctan(\sqrt{(4-c)/c})$. Similarly, for $a = \sigma e^{i\theta}$ and $\overline{a} = \sigma e^{i(-\theta)}$ we obtain $\sigma = D/\sqrt{4-c}$ and $\theta = -\arctan(\sqrt{c/(4-c)})$. Hence,

$$\begin{aligned} y_k = a\lambda^k + \overline{a}\overline{\lambda}^k &= \sigma\rho^k e^{i(k\varphi+\theta)} + \sigma\rho^k e^{-i(k\varphi+\theta)} \\ &= 2\sigma\rho^k \cos(k\varphi + \theta) \\ &= \frac{2D}{\sqrt{c^k(4-c)}} \cos\left(k \arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right). \end{aligned}$$

If we visualize the above equation in the complex plane, then y_k is the projection of the vector of $2a\lambda^k$ onto the x-axis. If we multiply two complex numbers, then the radii are multiplied and the angles are added. Hence, the sequence $2a\lambda^k$ turns by an angle of φ towards the second quadrant with each iteration. Once $2a\lambda^k$ is in the second quadrant, $2Re(a\lambda^k)$ is negative. This is illustrated in Figure 3 (see also [Hip94, IKL97, Kle97]).

Hence, y_k becomes negative as soon as there is an integer $l \geq 0$ with

$$k \arctan\sqrt{(4-c)/c} - \arctan\sqrt{c/(4-c)} \in (\pi/2 + l\pi, 3\pi/2 + l\pi).$$

Note that since $\arctan(x) < \pi/2$, we can choose $l = 0$ in the above expression and there is a k with

$$k \arctan\sqrt{(4-c)/c} - \arctan\sqrt{c/(4-c)} \in (\pi/2, 3\pi/2).$$

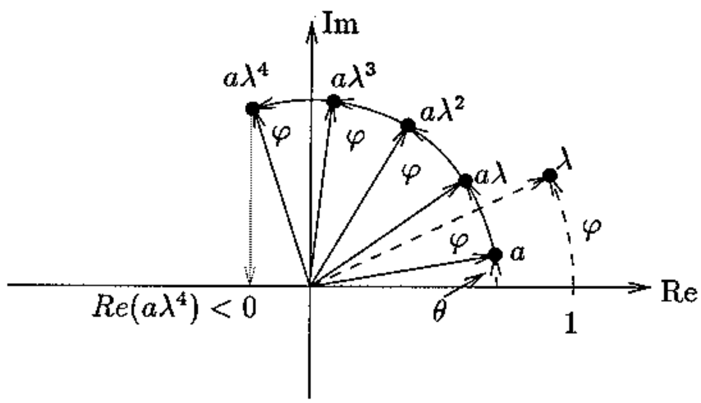

Figure 3: The sequence $2a\lambda^k$ turns by an angle of φ towards the second quadrant with each iteration.

We show that D can be chosen large enough such that $y_{n+1} < 0$ and $y_n > c$ or $y_{n-1}/y_n > c$. In the first case Lemma 3.6 implies that there is no strategy to search on the real line for a target at a distance at most D with a competitive ratio of $1+2c$. In the second case we note that the competitive ratio in the second step of strategy X is given by $1 + 2(y_n + y_{n-1})/y_n > 1 + 2c$ and the same claim follows.

Of course, we are interested in the smallest D for which the above inequalities holds. In the following we assume that $c \geq 3$.

Let n_0 be the first index such that $y_{n_0} < 0$, that is,

$$\cos\left(n_0 \arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right) < 0$$

or

$$n_0 = \left\lceil \frac{\arctan\left(\sqrt{\frac{c}{4-c}}\right) + \frac{\pi}{2}}{\arctan\left(\sqrt{\frac{4-c}{c}}\right)} \right\rceil .$$

We make two observations about n_0.

1. If $c \geq 3$, then we have

$$n_0 = \left\lceil \frac{\arctan\left(\sqrt{\frac{c}{4-c}}\right) + \frac{\pi}{2}}{\arctan\left(\sqrt{\frac{4-c}{c}}\right)} \right\rceil \leq \frac{\pi/2 + \pi/2}{3/4\sqrt{(4-c)/c}} \leq \frac{4\pi}{3}\sqrt{\frac{c}{4-c}} \leq \frac{9}{\sqrt{4-c}}. \quad (0.7)$$

   The first inequality stems from the fact that

   (a) $c \geq 3$, that is, $\sqrt{(4-c)/c} \leq 1/\sqrt{3}$ and

(b) $\arctan(x)' = 1/(1+x^2)$, that is, $\arctan(x) \geq x/(1+x^2)$ since arcus tangens is concave on the positive axis. Hence, $\arctan(\sqrt{(4-c)/c}) \geq \sqrt{(4-c)/c}/(1+\sqrt{1/3}^2)$.

2. Since n_0 is the smallest k such that $y_k < 0$,

$$(n_0 - 2)\arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right) \leq \frac{\pi}{2} - \arctan\left(\sqrt{\frac{4-c}{c}}\right). \quad (0.8)$$

W.l.o.g. we assume that y_{n_0} belongs to ray r_1. Since the search alternates between the two rays, the last point visited on ray r_1 has a distance of

$$\begin{aligned} y_{n_0-2} &= \frac{2D}{\sqrt{c^{n_0-2}(4-c)}}\cos\left((n_0-2)\arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right) \\ &\overset{(0.8)}{\geq} \frac{2D}{\sqrt{c^{n_0-2}(4-c)}}\cos\left(\frac{\pi}{2} - \arctan\left(\sqrt{\frac{4-c}{c}}\right)\right) \end{aligned} \quad (0.9)$$

to the origin. Since

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \arctan\left(\sqrt{\frac{4-c}{c}}\right)\right) &= \sin\left(\arctan\left(\sqrt{\frac{4-c}{c}}\right)\right) = \frac{\sqrt{(4-c)/c}}{\sqrt{1+(4-c)/c}} \\ &= \frac{\sqrt{c}}{2}\sqrt{\frac{4-c}{c}} = \frac{\sqrt{4-c}}{2}, \end{aligned} \quad (0.10)$$

we have

$$y_{n_0-2} \overset{(0.9,0.10)}{\geq} \frac{2D}{\sqrt{c^{n_0-2}(4-c)}}\frac{\sqrt{4-c}}{2} = \frac{D}{\sqrt{c^{n_0-2}}} \overset{(0.7)}{\geq} \frac{D}{\sqrt{c^{9/\sqrt{4-c}}}}.$$

Proposition 4.1 *If* $3 < c < 4 - 81/\log^2(D/16)$, *then* $D/\sqrt{c^{9/\sqrt{4-c}}} > c^2$.

Proof: We have

$$\begin{array}{rclcrcll} c &<& 4 - \dfrac{81}{\log^2(D/16)} & \Rightarrow & 4-c &>& \dfrac{81}{\log^2(D/16)} & \overset{(\log c<2)}{\Rightarrow} \\ \log D &>& \left(\dfrac{4.5}{\sqrt{4-c}} + 2\right)\log c & \Rightarrow & 2\log D &>& \left(\dfrac{9}{\sqrt{4-c}} + 4\right)\log c & \Rightarrow \\ D^2 &>& c^{9/\sqrt{4-c}+4} & \Rightarrow & \dfrac{D}{\sqrt{c^{9/\sqrt{4-c}}}} &>& c^2 & \end{array}$$

. □

Let $3 < c < 4 - 81/\log^2(D/16)$. Proposition 4.1 implies that $y_{n_0-2} > c^2$ and $y_{n_0} < 0$. Hence, if $y_{n_0-1} \leq c$, then $(y_{n_0-1} + y_{n_0-2})/y_{n_0-1} > c$; otherwise $y_{n_0-1} > c$. Therefore, Y satisfies Theorem 3.5.

Finally, we also consider the case that $c \in [1,3]$; then, $n_0 \leq \pi/\arctan(1/3) = 6$ and Equations 0.9 and 0.10 still hold. Hence,

$$\begin{aligned} y_{n_0-2} &= \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \cos\left((n_0-2)\arctan\left(\sqrt{\frac{4-c}{c}}\right) - \arctan\left(\sqrt{\frac{c}{4-c}}\right)\right) \\ &\geq \frac{2D}{\sqrt{c^{n_0-2}(4-c)}} \frac{\sqrt{4-c}}{2} \geq \frac{D}{\sqrt{c^4}} \geq \frac{D}{9}. \end{aligned}$$

Hence, if $D > 81$, then $y_{n_0-2} > 9$ and $y_{n_0} < 0$ and there is no strategy with a competitive ratio of less than or equal to $1+2\cdot 3 = 7$. Note that this also holds for strategies to $m > 2$ rays since any such strategy can be used to search on two rays and the competitive ratio only improves.

5 Solving the Recurrence Equation for the General Case

We now return to the general case. As for the case $m = 2$ we want to show that if there are only complex or negative solutions to Equation 0.5, then the angle of the polar coordinates of the solutions turn towards a negative solution. However, the details are much more complicated than in the case $m = 2$ since we have many roots of Equation 0.5 and the solutions cannot be computed explicitly. One possibility to get around this problem is to use estimates on the angles and radii of the roots. We show that there is one root λ which has the largest radius among all roots of Equation 0.5. After a sufficiently large number of steps the contribution of λ dominates the contribution of all other solutions.

Let $\lambda_0, \dots, \lambda_{m-1}$ be the roots of Equation 0.5. The solution of the recurrence is given by

$$y_k = a_0\lambda_0^k + a_0\lambda_0^k + \cdots + a_{m-1}\lambda_{m-1}^k.$$

We first investigate the structure of the roots λ_i, $0 \leq i \leq m-1$.

Let λ be a complex root of Equation 0.5. We consider the polar coordinates of λ, that is, we set $\lambda = \rho e^{i\varphi}$.

Lemma 5.1 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 0.5, then $\rho = \sin(m-1)\varphi/\sin m\varphi$.*

Proof: Let $\lambda = \rho e^{i\varphi}$ be a complex root of Equation 0.5. We have $\lambda^{m-1} = \rho^{m-1}e^{i(m-1)\varphi}$ and

$$\begin{aligned} \lambda^{m-1}(\lambda - 1) &= \rho^{m-1}\left(\cos(m-1)\varphi + i\sin(m-1)\varphi\right)\left(\rho\cos\varphi - 1 + i\rho\sin\varphi\right) \\ &= \rho^{m-1}\left(\cos(m-1)\varphi(\rho\cos\varphi - 1) - \sin(m-1)\varphi\rho\sin\varphi + \right. \\ &\quad \left. i(\cos(m-1)\varphi\rho\sin\varphi + \sin(m-1)\varphi(\rho\cos\varphi - 1))\right) \\ &= \rho^{m-1}\left(\cos(m-1)\varphi\rho\cos\varphi - \sin(m-1)\varphi\rho\sin\varphi - \cos(m-1)\varphi + \right. \\ &\quad \left. i(\cos(m-1)\varphi\rho\sin\varphi + \sin(m-1)\varphi\rho\cos\varphi - \sin(m-1)\varphi)\right) \\ &= \rho^{m-1}\left(\rho\cos m\varphi - \cos(m-1)\varphi + i(\rho\sin m\varphi - \sin(m-1)\varphi)\right). \end{aligned}$$

Since $\lambda^{m-1}(\lambda-1) = -1/c \in I\!R$, we obtain

$$\rho \sin m\varphi - \sin(m-1)\varphi = 0 \qquad \text{or} \qquad \rho = \frac{\sin(m-1)\varphi}{\sin m\varphi} \tag{0.11}$$

as claimed. □

Lemma 5.1 has the following consequence.

Corollary 5.2 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 0.5, then λ is solely determined by φ.*

5.1 The Polar Angle of a Root

We first concentrate on the polar angle of a root λ of Equation 0.5.

Lemma 5.3 *If $\lambda = \rho e^{i\varphi}$ is a complex root of Equation 0.5, then $\varphi \in [2k\pi/(m-1), (2k+1)\pi/m]$.*

Proof: Let $\lambda = \rho e^{i\varphi}$ be a complex root of Equation 0.5. Equation 0.11 implies that $m\varphi$ and $(m-1)\varphi$ either *both* belong to $[2k\pi, (2k+1)\pi]$ or to $[(2k+1)\pi, (2k+2)\pi]$ as $\rho > 0$ and, therefore, both $\sin m\varphi$ and $\sin(m-1)\varphi$ have the same sign. Since $-1/c = \lambda^{m-1}(\lambda-1) = \rho^{m-1}(\rho\cos(m\varphi) - \cos(m-1)\varphi) < 0$ and $\rho^{m-1} > 0$, we also need $\frac{\sin(m-1)\varphi}{\sin m\varphi}\cos m\varphi - \cos(m-1)\varphi < 0$ as $\rho = \frac{\sin(m-1)\varphi}{\sin m\varphi}$. If $\varphi \in (2k\pi/(m-1), (2k+1)\pi/m]$, that is, $\sin(m-1)\varphi > 0$, then $\cot(m-1)\varphi > \cot m\varphi$ and the above inequality holds. If $\varphi \in ((2k+1)\pi/(m-1), (2k+2)\pi/m]$, that is, $\sin(m-1)\varphi < 0$, then we need $\cot(m-1)\varphi < \cot m\varphi$ which is impossible since cotangens decreases monotonely over $[\pi, 2\pi]$. Hence, $\varphi \in [2k\pi/(m-1), (2k+1)\pi/m]$, for $0 \le k \le \lfloor m/2 \rfloor$ as claimed. □

We first show that there is exactly one root λ_k for each interval $[2k\pi/(m-1), (2k+1)\pi/m]$ with $0 \le k \le \lfloor m/2 \rfloor$.

Lemma 5.4 *For each interval $[2k\pi/(m-1), (2k+1)\pi/m]$ with $0 \le k \le \lfloor m/2 \rfloor$, there is exactly one root $\lambda_k = \rho_k e^{i\varphi_k}$ of Equation 0.5 with $\varphi_k \in [2k\pi/(m-1), (2k+1)\pi/m]$.*

Proof: Since λ is a function of φ by Corollary 5.2, it suffices to show that $1/(\lambda^{m-1}(\lambda-1))$ is monotone in φ and that $1/(\lambda^{m-1}(1-\lambda))$ assumes a value less than and greater than c, for each interval $[2k\pi/(m-1), (2k+1)\pi/m]$ with $0 \le k \le \lfloor m/2 \rfloor$. We first show that $1/(\lambda^{m-1}(\lambda-1))$ is monotone in φ.

$$\begin{aligned}
\frac{\partial}{\partial\varphi}\frac{1}{\lambda^{m-1}(\lambda-1)} &= \frac{\partial}{\partial\varphi}\frac{\sin^{m-1} m\varphi}{\sin^m(m-1)\varphi(\cot(m-1)\varphi - \cot m\varphi)} \\
&= -\frac{\sin^{m-1} m\varphi}{\sin^m(m-1)\varphi}\frac{m^2\sin^2\varphi + \sin^2 m\varphi - 2m\sin\varphi\sin m\varphi\cos(m-1)\varphi}{\sin^2\varphi} \\
&\le 0
\end{aligned}$$

since

$$\begin{aligned}
& m^2\sin^2\varphi + \sin^2 m\varphi - 2m\sin\varphi\sin m\varphi\cos(m-1)\varphi \\
\geq\ & m^2\sin^2\varphi + \sin^2 m\varphi - 2m|\sin\varphi||\sin m\varphi| \\
=\ & (m|\sin\varphi| - |\sin m\varphi|)^2 \geq 0.
\end{aligned}$$

Hence, there is at most one root of Equation 0.5 for every interval $[2k\pi/(m-1), (2k+1)\pi/m]$, for $0 \leq k \leq \lfloor m/2 \rfloor - 1$. Since $\sin^{m-1} m\varphi/(\sin^m(m-1)\varphi(\cot(m-1)\varphi - \cot m\varphi))$ is continuous over $[2k\pi/(m-1), (2k+1)\pi/m]$ and its values range from ∞ to 0, there is also at least one root of of Equation 0.5 with a polar angle in $[2k\pi/(m-1), (2k+1)\pi/m]$, for $0 \leq k \leq \lfloor m/2 \rfloor - 1$. □

The above roots account for $\lfloor m/2 \rfloor$ roots of Equation 0.5. If m is odd, then there is one root $\lambda_{\lfloor m/2 \rfloor}$ with $\varphi_{\lfloor m/2 \rfloor} = 2\lfloor m/2 \rfloor \pi/(m-1) = (2\lfloor m/2 \rfloor + 1)\pi/m = \pi$, that is, $\lambda_{\lfloor m/2 \rfloor}$ is a negative real root. The remaining $\lfloor m/2 \rfloor$ roots are given by the conjugates $\overline{\lambda}_k = \rho_k e^{-i\varphi_k}$ of λ_k as in the case $m = 2$ since we have that if λ is a root of Equation 0.5, then

$$\begin{aligned}
\overline{\lambda}^{m-1}(\overline{\lambda} - 1) &= \rho^{m-1} e^{-im\varphi}(\rho e^{-i\varphi} - 1) \\
&= \rho^{m-1}(\rho\cos(-m\varphi) - \cos(-(m-1)\varphi) \\
&\quad + i(\rho\sin(-m\varphi) - \sin(-(m-1)\varphi))) \\
&= \rho^{m-1}(\rho\cos m\varphi - \cos(m-1)\varphi + i(\sin(m-1)\varphi - \rho\sin m\varphi)) \\
&= \rho^{m-1}(\rho\cos m\varphi - \cos(m-1)\varphi) = -\frac{1}{c}
\end{aligned}$$

since $\sin(m-1)\varphi - \rho\sin m\varphi = 0$. Hence, if λ is a root of Equation 0.5, then $\overline{\lambda}$ is also a root of Equation 0.5.

Let φ_k be the angle of the root in $[2k\pi/(m-1), (2k+1)\pi/m]$. In the following we calculate a lower bound on the size of φ_0 if $c < m^m/(m-1)^{m-1}$.

Lemma 5.5

$$\varphi_0 \geq \min\left\{\frac{1}{m^{3/2}}\sqrt{\frac{m^m}{(m-1)^{m-1}} - c}, \frac{1}{\sqrt{3}m}\right\}.$$

Proof: We assume that $\varphi_0 \in [0, \pi/\sqrt{3}m]$ since if $\varphi_0 \geq \pi/\sqrt{3}m$, then the claim trivially holds.

$$
\begin{aligned}
c = \frac{1}{\lambda_0^{m-1}(1-\lambda_0)} &= \left(\frac{\sin m\varphi_0}{\sin(m-1)\varphi_0}\right)^{m-1} \frac{1}{\sin(m-1)\varphi_0} \frac{1}{\cot(m-1)\varphi_0 - \cot m\varphi_0} \\
&= \left(\frac{\sin m\varphi_0}{\sin(m-1)\varphi_0}\right)^{m-1} \frac{1}{\sin(m-1)\varphi_0} \frac{\sin m\varphi_0 \sin(m-1)\varphi_0}{\sin\varphi_0} \\
&\geq \left(1 + \frac{(m - m^3\varphi_0^2/6)\varphi_0}{(m-1)\varphi_0}\right)^{m-1} \frac{\sin m\varphi_0}{\sin\varphi_0} \\
&\geq \left(1 + \frac{m - m^3\varphi_0^2/6}{(m-1)}\right)^{m-1} \frac{m\varphi_0 - (m\varphi_0)^3/6}{\varphi_0} \\
&\geq \left(1 + \frac{m - m^3\pi^2/(18m^2)}{(m-1)}\right)^{m-1} \frac{m\varphi_0 - (m\varphi_0)^3/6}{\varphi_0} \qquad (\varphi_0 \leq \pi/\sqrt{3}m) \\
&\geq \left(\frac{m-1+m-m\pi^2/18}{m-1}\right)^{m-1} \left(m - \frac{m^3\varphi_0^2}{6}\right) \\
&\geq \left(\frac{m}{m-1}\right)^{m-1} \left(m - \frac{m^3\varphi_0^2}{6}\right) \\
&\geq \left(1 - \frac{m^2\varphi_0^2}{6}\right) \frac{m^m}{(m-1)^{m-1}}.
\end{aligned}
$$

Here we use that by the Taylor-expansion of $\sin x - x^3/6 \leq \sin(x) \leq x$ if $x \geq 0$. Since $m^m/(m-1)^{m-1} < em$, we have

$$
\varphi_0 \geq \min\left\{\sqrt{\frac{6(m^m/(m-1)^{m-1} - c)}{em^3}}, \frac{1}{\sqrt{3}m}\right\} \geq \min\left\{\frac{1}{m^{3/2}}\sqrt{\frac{m^m}{(m-1)^{m-1}} - c}, \frac{1}{\sqrt{3}m}\right\} \tag{0.12}
$$

as claimed. □

5.2 The Radius of a Root

We now consider the radius of a root of Equation 0.5. Let ρ_k be the radius of λ_k. In the following we show that $\rho_k \geq \rho_{k+1}$, for all $0 \leq k \leq \lceil m/2 \rceil - 1$.

Lemma 5.6 *For all* $0 \leq k \leq \lceil m/2 \rceil - 1$, $\rho_k \geq \rho_{k+1}$.

Proof: We first observe that

$$
\lambda_k^{m-1}(1-\lambda_k) = |\lambda_k^{m-1}(1-\lambda_k)| = \rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1}.
$$

We show that $\rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1}$ is monotonely increasing in ρ_k.

We consider the derivative of $\rho^{m-1}\sqrt{\rho^2 - 2\rho\cos\varphi + 1}$ with respect to ρ and obtain

$$\frac{\partial}{\partial\rho}\rho^{m-1}\sqrt{\rho^2 - 2\rho\cos(\varphi) + 1} = \frac{\rho^{m-2}m(\rho^2 - (2m-1)/m\rho\cos\varphi + (m-1)/m)}{\sqrt{\rho^2 - 2\rho\cos\varphi + 1}}.$$

Hence, $\rho^{m-1}\sqrt{\rho^2 - 2\rho\cos\varphi + 1}$ has an extremum greater than zero with respect to ρ if

$$\rho^2 - 2\rho\cos\varphi + 1 - (1 - \rho\cos\varphi)/m = 0,$$

that is, if

$$\rho = \frac{1}{2}\frac{(2m-2)\cos\varphi \pm \sqrt{1 - ((2m-1)\sin\varphi)^2}}{m}. \qquad (0.13)$$

The above equation implies that

$$\sin\varphi \leq \frac{1}{2m-1}$$

or $\varphi \leq \arcsin(1/2m-1) \leq 2/(m-1) < 2\pi/(m-1)$, for $m \geq 3$, as $\sin(x) \geq 1/(2x)$, for $0 \leq x \leq \pi/3$. Therefore, the root of Equation 0.13 is outside the interval $[2k\pi/(m-1), (2k+1)\pi/m]$ and $\rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1}$ is monotonely increasing in ρ_k, for all $1 \leq k \leq \lceil m/2 \rceil$, but not for $k = 0$. We now show that this implies that $\rho_0 \geq \rho_1 \geq \cdots \geq \rho_{\lceil m/2 \rceil}$. Let $0 \leq k \leq \lceil m/2 \rceil - 1$. Since $\varphi_{k+1} > \varphi_k$, we have, for $0 \leq k \leq \lceil m/2 \rceil - 1$,

$$\rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_k + 1} < \rho_k^{m-1}\sqrt{\rho_k^2 - 2\rho_k\cos\varphi_{k+1} + 1}$$

and as $\rho_{k+1}^{m-1}\sqrt{\rho_{k+1}^2 - 2\rho_{k+1}\cos\varphi_{k+1} + 1}$ is monotone in ρ_{k+1}, ρ_{k+1} has to be decreased in order to obtain equality. □

In the following we investigate the ratio ρ_0/ρ_k.

Lemma 5.7 *$\rho_0/\rho_k \geq 1 + 1/(4m^3)$, for all $1 \leq k \leq \lceil m/2 \rceil$.*

Proof: Since by Lemma 5.6 $\rho_1 \geq \rho_k$, for all for all $2 \leq k \leq \lceil m/2 \rceil$, it suffices to show that $\rho_0/\rho_1 \geq 1 + 1/(4m^3)$. Let f be the function

$$f(\varphi, \rho) = \left|\lambda^{m-1}(1-\lambda)\right|^2 = \rho^{2(m-1)}(\rho^2 - 2\rho\cos\varphi + 1).$$

Note that $f(\varphi_0, \rho_0) = f(\varphi_1, \rho_1) = 1/c^2$ and, therefore,

$$f(\varphi_1, \rho_0) - f(\varphi_0, \rho_0) = f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1).$$

Now

$$f(\varphi_1, \rho_0) - f(\varphi_0, \rho_0) = 2\rho_0^{2m-1}(\cos\varphi_0 - \cos\varphi_1)$$

and

$$f(\varphi_1, \rho_0) - f(\varphi_1, \rho_1) = \int_{\rho_1}^{\rho_0} \frac{\partial}{\partial\rho} f(\varphi_1, \rho) d\rho \leq (\rho_0 - \rho_1) \max_{\rho \in [\rho_1, \rho_0]} \frac{\partial}{\partial\rho} f(\varphi_1, \rho).$$

If we consider the derivative of f with respect to ρ, then

$$\begin{aligned}\frac{\partial}{\partial\rho}f(\varphi_1,\rho) &= 2m\rho^{2m-1} - 2(2m-1)\rho^{2(m-1)}\cos\varphi + 2(m-1)\rho^{2m-3} \\ &= 2m\rho^{2m-3}\left(\rho^2 - 2\frac{2m-1}{2m}\rho\cos\varphi + \frac{2(m-1)}{2m}\right).\end{aligned}$$

Hence,

$$f(\varphi_1,\rho_0) - f(\varphi_1,\rho_1) \;\leq\; (\rho_0-\rho_1)\max_{\rho\in[\rho_1,\rho_0]} 2m\rho^{2m-3}\left(\rho^2 - \frac{2m-1}{m}\rho\cos\varphi + \frac{2(m-1)}{2m}\right).$$

If we add $(2+(2m-1)/m)\rho\cos\varphi + 1 - (2m-1)/(2m)$ to the term in the paranthesis, then

$$f(\varphi_1,\rho_0) - f(\varphi_1,\rho_1) \;\leq\; (\rho_0-\rho_1)2m\rho_0^{2m-3}(\rho_0+1)^2$$

and

$$\begin{aligned}2\rho_0^{2m-1}(\cos\varphi_0 - \cos\varphi_1) &\leq (\rho_0-\rho_1)2m\rho_0^{2m-3}(\rho_0+1)^2 \qquad \Longleftrightarrow \\ \frac{\rho_0}{\rho_1}\frac{\rho_0(\cos\varphi_0-\cos\varphi_1)}{m(\rho_0+1)^2} &\leq \frac{\rho_0}{\rho_1} - 1\end{aligned}$$

or

$$\frac{\rho_0}{\rho_1} \;\geq\; \frac{1}{1-\rho_0(\cos\varphi_0-\cos\varphi_1)/(m(\rho_0+1)^2)} \geq 1 + \frac{\rho_0(\cos\varphi_0-\cos\varphi_1)}{m(\rho_0+1)^2}.$$

In order to bound $\rho_0(\cos\varphi_0 - \cos\varphi_1)/(m(\rho_0+1)^2)$ from below, we need upper and lower bounds for ρ_0. We first give an upper bound. Observe that

$$\begin{aligned}&\lambda^{m-1}(1-\lambda) = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^{m-1}\frac{\sin\varphi_0}{\sin m\varphi_0} = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^{m}\frac{\sin\varphi_0}{\sin(m-1)\varphi_0} = \frac{1}{c} \\ &\Rightarrow \rho_0^m = \left(\frac{\sin(m-1)\varphi_0}{\sin m\varphi_0}\right)^m = \frac{\sin(m-1)\varphi_0}{\sin\varphi_0 c} \leq \frac{m-1}{c}.\end{aligned}$$

Hence, $\rho_0 \leq \sqrt[m]{(m-1)/c} \leq 1$ since $c \geq 3$.

Now note that $|1-\lambda_0|$ is the distance between the point $(1,0)$ and the point λ_0 in the complex plane. Since λ_0 belongs to the wedge S_0 of numbers whose polar angle is in $[0,\pi/3]$ and whose radius is less than one, it is easy to see that the origin is the furthest point in S_0 from $(1,0)$ and $|1-\lambda_0| \leq 1$. Hence, $\rho_0 \geq \sqrt[m-1]{1/(|1-\lambda_0|c)} \geq \sqrt[m-1]{1/c}$. Since we assume that $c < m^m/(m-1)^{m-1} < em$, we obtain, $\rho_0 \geq \sqrt[m-1]{1/(em)} \geq 1/3$.

Next we give a lower bound for $\cos\varphi_0 - \cos\varphi_1$. Since $\varphi_0 \in [0,\pi/m]$ and $\varphi_1 \in [2\pi/(m-1), 3\pi/m]$ both of which are contained in $[0,\pi]$, for $m \geq 3$, $\cos\varphi_0 - \cos\varphi_1 \geq \cos\pi/m - \cos 2\pi/(m-1)$. Moreover, since cosine is concave over $[0,\pi/2]$ and $2\pi/(m-1) \leq \pi/2$, for $m \geq 5$,

$$\cos\varphi_0 - \cos\varphi_1 \geq \cos\frac{\pi}{m} - \cos\frac{2\pi}{m-1} \geq \sin\frac{\pi}{m}\left(\frac{2\pi}{m-1} - \frac{\pi}{m}\right) \geq \frac{\pi}{2m}\frac{\pi}{m} \geq \frac{\pi^2}{2m^2},$$

for $m \geq 5$. On the other hand, if $m = 3$, then $\cos(\pi/3) - \cos(2\pi/2) > 1 > \pi^2/18$ and if $m = 4$, then $\cos(\pi/4) - \cos(2\pi/3) > 1/\sqrt{2} > \pi^2/32$, so that the inequality $\cos\varphi_0 - \cos\varphi_1 \geq \pi^2/(2m^2)$ holds for all $m \geq 3$.

Hence, for $1 \leq k \leq \lceil m/2 \rceil$,

$$\frac{\rho_0}{\rho_k} \geq \frac{\rho_0}{\rho_1} \ \geq \ 1 + \frac{\pi^2}{6m^3(1+1)^2} \geq 1 + \frac{1}{4m^3}.$$

□

5.3 The Coefficients

We finally give an upper bound on the radius of the coefficients. Recall that the solution of Recurrence Equation 0.5 is given by

$$y_k = a_0\lambda_0^k + a_1\lambda_1^k + \cdots + a_{m-1}\lambda_{m-1}^k.$$

The coefficients a_i are the solution of the linear equation system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{m-1} & \lambda_1^{m-1} & \cdots & \lambda_{m-1}^{m-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} D \\ D \\ \vdots \\ D \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_0 & \lambda_1 & \cdots & \lambda_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{m-1} & \lambda_1^{m-1} & \cdots & \lambda_{m-1}^{m-1} \end{pmatrix}$$

and $A_i(x)$ the matrix A where the ith column is replaced by the vector $(x, \ldots, x)^T$. By Cramer's rule a_i is given as

$$\begin{aligned} a_i &= \det(A_i(D))/\det(A) = D\det(A_i(1))/\det(A) \\ &= D\frac{\prod_{j=0,j\neq 0}^{m-1}(1-\lambda_j)}{\prod_{j=0,j\neq i}^{m-1}(\lambda_i - \lambda_j)} \end{aligned} \qquad (0.14)$$

since both A and $A_i(1)$ are Vandermonde matrices with

$$\begin{aligned} \det(A_i(1)) &= \prod_{j=0,j\neq i}^{m-1}(1-\lambda_j) \prod_{j<k,j,k\neq i}(\lambda_k - \lambda_j) \qquad \text{and} \\ \det(A) &= \prod_{j=0,j\neq i}^{m-1}(\lambda_i - \lambda_j) \prod_{j<k,j,k\neq i}(\lambda_k - \lambda_j). \end{aligned}$$

In order to bound the size of the ratio of $|a_i/a_0|$ we have the following lemma.

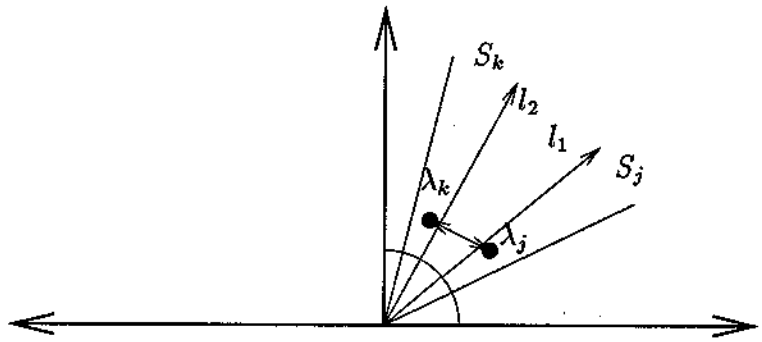

Figure 4: The sectors that λ_k and λ_j belong to.

Lemma 5.8

$$\left|\frac{a_i}{a_0}\right| \leq 4^{2m}m^m.$$

Proof: We have

$$\begin{aligned}\left|\frac{a_i}{a_0}\right| &= \left|\frac{1-\lambda_i}{1-\lambda_0}\right|\left|\frac{\prod_{j=0,j\neq 0}^{m-1}(\lambda_0-\lambda_j)}{\prod_{j=0,j\neq i}^{m-1}(\lambda_i-\lambda_j)}\right| \\ &\leq \frac{1+|\lambda_i|}{|1-\lambda_0|}\frac{\prod_{j=0,j\neq 0}^{m-1}(|\lambda_0|+|\lambda_j|)}{\prod_{j=0,j\neq i}^{m-1}|\lambda_i-\lambda_j|} \\ &\leq \frac{2}{|1-\lambda_0|}\frac{2^{m-1}}{\prod_{j=0,j\neq i}^{m-1}|\lambda_i-\lambda_j|}\end{aligned}$$

In order to obtain a lower bound for $|1-\lambda_0|$ we observe that

$$|1-\lambda_0| = \frac{1}{c|\lambda_0^{m-1}|} \geq \frac{1}{c} \geq \frac{1}{em} \tag{0.15}$$

Finally, we give a lower bound for $|\lambda_k - \lambda_j|$. Since $|1-\lambda_i| \leq 1+|\lambda_i| \leq 2$, $\lambda_i^{m-1} \geq 1/(2c) \geq 1/(2em)$ or $\lambda_i \geq \sqrt[m-1]{1/(2em)} \geq 1/5$.

If we view λ_k and λ_j as two points in the complex plane, then λ_k is contained in the angular sector of $S_k = [2k\pi/(m-1), (2k+1)\pi/m]$ and λ_j is contained in the angular sector of $S_j = [2j\pi/(m-1), (2j+1)\pi/m]$ (see Figure 4). Since $|\lambda_k| \geq 1/5$ and $|\lambda_j| \geq 1/5$, the distance between λ_k and λ_j is at least the distance between the points of S_k and S_j outside the circle with radius $1/5$. W.l.o.g. assume that $k > j$. Let l_1 be the line with angle $2k\pi/(m-1)$ through the origin and l_2 be the line with angle $(2j+1)\pi/m$ through the origin. If p is the point on l_1 with distance $1/5$ to the origin, then the distance of S_k to S_j outside the circle with radius $1/5$ is at most the distance of p to l_2. By elementary geometry we obtain that

$$|\lambda_k - \lambda_j| \geq d(p, l_2) = \frac{\sin\left(2k\pi/(m-1) - (2j+1)\pi/m\right)}{5} \geq \frac{\pi}{10m} \geq \frac{1}{4m}. \tag{0.16}$$

Combining the estimates for $|1-\lambda_0|$ and $|\lambda_k - \lambda_j|$ we obtain

$$\left|\frac{a_i}{a_0}\right| \leq \frac{2^m}{|1-\lambda_0|\prod_{j=0,j\neq i}^{m-1}|\lambda_i-\lambda_j|} \leq 2^m em(4m)^{m-1} \leq 4^{2m}m^m$$

as claimed. □

The following lemma gives a lower bound of the absolute value of a_0.

Lemma 5.9

$$|a_0| > \frac{D}{(2em)^{m-1}}.$$

Proof: The proof follows easily from Equations 0.14 and 0.15.

$$|a_0| = D\frac{\prod_{j=1}^{m-1}|1-\lambda_j|}{\prod_{j=1}^{m-1}|\lambda_0-\lambda_j|} \geq D\frac{(1/em)^{m-1}}{2^{m-1}}.$$

Note that the lower bound for $|1-\lambda_0|$ of Equation 0.15 is also a lower bound for $|1-\lambda_i|$ and that $|\lambda_0 - \lambda_j| \leq \rho_0 + \rho_j < 2$. □

5.4 Putting it all Together

We now put the estimates we obtained for the radii and the angles of the roots of Equation 0.5 as well as the coefficients into use. W.l.o.g. we assume that m is even. If m is odd an analogous proof works. We start off by proving a lower and an upper bound on the size of y_k.

Lemma 5.10

$$y_k \leq 2|a_0|\rho_0^k\left(\cos(\theta_0+k\varphi_0)+\frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^k}\right).$$

and

$$y_k \geq 2|a_0|\rho_0^k\left(\cos(\theta_0+k\varphi_0)-\frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^k}\right).$$

Proof: Recall that

$$y_k = \sum_{j=0}^{\lfloor m/2\rfloor} a_j\lambda_j^k + \overline{a}_j\overline{\lambda}_j^k \leq a_0\lambda_0^k + \overline{a}_0\overline{\lambda}_0^k + \sum_{j=0}^{\lfloor m/2\rfloor} 2|a_j\lambda_j^k|.$$

If $\lambda_0 = \rho_0 e^{i\varphi_0}$ and $a_0 = \sigma_0 e^{i\theta_0}$, then

$$a_0\lambda_0^k + \overline{a}_0\overline{\lambda}_0^k = \sigma_0\rho_0^k e^{i(\theta_0+k\varphi_0)} + \sigma_0\rho_0^k e^{-i(\theta_0+k\varphi_0)} = 2\sigma_0\rho_0^k\cos(\theta_0+k\varphi_0).$$

and

$$\begin{aligned} y_k &\le 2|a_0|\rho_0^k\left(\cos(\theta_0+k\varphi_0)+\sum_{j=0}^{\lfloor m/2\rfloor}\left|\frac{a_j}{a_0}\right|\left|\frac{\rho_j^k}{\rho_0^k}\right|\right)\\ &\le 2|a_0|\rho_0^k\left(\cos(\theta_0+k\varphi_0)+\frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^k}\right)\end{aligned}$$

by Lemmas 5.7 and 5.8. Similarly,

$$\begin{aligned} y_k &\ge 2|a_0|\rho_0^k\left(\cos(\theta_0+k\varphi_0)-\sum_{j=0}^{\lfloor m/2\rfloor}\left|\frac{a_j}{a_0}\right|\left|\frac{\rho_j^k}{\rho_0^k}\right|\right)\\ &\ge 2|a_0|\rho_0^k\left(\cos(\theta_0+k\varphi_0)-\frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^k}\right).\end{aligned}$$

□

We claim that if

$$c<\frac{m^m}{(m-1)^{m-1}}-\frac{22^2m^8\log^2 m}{\log^2 D},$$

then there is a step k such that $y_k>c^2$ and $y_{k+2}<0$, that is, there is no strategy X such that the competitive ratio of X is $1+2c$ and all the points in $[c,D]$ are searched by X for all rays r_j, $0\le j\le m-1$.

In the following let $\varepsilon=\sqrt{m^m/(m-1)^{m-1}-c}$. We assume that $\varepsilon<1$. The case $\varepsilon\ge 1$ can be treated as the case $c\le 3$ in the case $m=2$.

Let k_0 be the first index greater than $4m^3(3m\log m-\log\varepsilon)+1$ such that

$$\cos(\theta_0+k_0\varphi_0)>0 \qquad\text{and}\qquad \cos(\theta_0+(k_0+1)\varphi_0)\le 0.$$

We show the following bounds on y_{k_0-1} and y_{k_0+2}.

Lemma 5.11

$$y_{k_0-1}\ge 2|a_0|\rho_0^{k_0-1}\frac{\varphi_0}{4} \qquad\textit{and}\qquad y_{k_0+2}\le -2|a_0|\rho_0^{k_0+2}\frac{\varphi_0}{4}.$$

Proof: We first observe that if $k_0>4m^3(3m\log m-\log\varepsilon)+1$, then

$$\begin{aligned} k_0-1 &\ge \frac{3m\log m-\log\varepsilon}{\log(1+1/(4m^3))} &&\text{(since } \log(1+x)\le x)\\ &\ge \frac{(m+1)\log m+\log(4m+2)+\log(m^{3/2}/\varepsilon)}{\log(1+1/(4m^3))}.\end{aligned}$$

Note that since $\varepsilon\le 1$, $\varepsilon/m^{3/2}<1/\sqrt{3}m$ and $\varphi_0\ge\varepsilon/m^{3/2}$ by Lemma 5.5 which implies that

$$\left(1+\frac{1}{4m^3}\right)^{k_0-1}\ge\frac{4^{2m+1}m^{m+1}}{\varphi_0} \qquad\text{and}\qquad \frac{4^{2m}m^{m+1}}{(1+1/4m^3)^{k_0-1}}\le\frac{\varphi_0}{4}.$$

In particular, if k_0 is the first index greater than $4m^3(3m\log m - \log\varepsilon)$ such that $\cos(\theta_0 + k_0\varphi_0) > 0$ and $\cos(\theta_0 + (k_0+1)\varphi_0) \le 0$, then by Lemma 5.10

$$\begin{aligned}
\frac{y_{k_0-1}}{2|a_0|\rho_0^{k_0-1}} &\ge \cos(\theta_0 + (k_0-1)\varphi_0) - \frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^{k_0-1}} \\
&\ge \cos(\pi/2 - \varphi_0) - \frac{\varphi_0}{4} = \sin(\varphi_0) - \frac{\varphi_0}{4} \\
&\ge \frac{\varphi_0}{2} - \frac{\varphi_0}{4} = \frac{\varphi_0}{4}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{y_{k_0+2}}{2|a_0|\rho_0^{k_0+2}} &\le \cos(\theta_0 + (k_0+2)\varphi_0) + \frac{4^{2m}m^{m+1}}{(1+1/(4m^3))^{k_0}} \\
&\le \cos(\pi/2 + \varphi_0) + \frac{\varphi_0}{4} = -\sin(\varphi_0) + \frac{\varphi_0}{4} \\
&\le -\frac{\varphi_0}{2} + \frac{\varphi_0}{4} = -\frac{\varphi_0}{4}
\end{aligned}$$

as claimed. □

We now bound the value of k_0. Since the distance between two consecutive transitions from positive to negative values of cosine is at most 2π and $k_0 \ge 4m^3(3m\log m - \log\varepsilon) + 1$, we have that $k_0 - 4m^3(3m\log m - \log\varepsilon) - 1 \le 2\pi/\varphi_0$ and by Lemma 5.5

$$k_0 \le 4m^3(3m\log m - \log\varepsilon) + 1 + \frac{2\pi}{\varphi_0} \le 4m^3(3m\log m - \log\varepsilon) + 1 + \frac{2\pi m^{3/2}}{\varepsilon}. \quad (0.17)$$

With the above preparations we now can prove the main lemma.

Lemma 5.12 *If*

$$c < \frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D},$$

then $y_{k_0-1} > c^2$ *and* $y_{k_0+2} < 0$.

Proof: $y_{k_0+2} < 0$ follows directly from Lemma 5.11. Hence, we only have to show $y_{k_0-1} > c^2$.

Step 1 We first show that if

$$c < \frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D},$$

then

$$D > \frac{c^2(2em)^m 3^{k_0}}{\varepsilon},$$

where again $\varepsilon = \sqrt{m^m/(m-1)^{m-1} - c}$. We note that if

$$c < \frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D},$$

then

$$\begin{aligned}
\log D &> \frac{22m^4 \log m}{\sqrt{m^m/(m-1)^{m-1} - c}} = \frac{22m^4 \log m}{\varepsilon} \\
&\geq (12m^4 \log m - 4m^3 \log \varepsilon) \log 3 + \frac{m^4 \log m}{\varepsilon} \qquad \text{(since } 1/\varepsilon > -\log \varepsilon) \\
&\geq \left(4m^3(3m \log m - \log \varepsilon) + \frac{2\pi m^{3/2}}{\varepsilon}\right) \log 3 + \\
&\quad (m-1)\log(2em) + 2\log(em) + \log m^{3/2}
\end{aligned}$$

and, therefore,

$$D > \frac{(em)^2(2em)^{m-1}3^{k_0-1}m^{3/2}}{\varepsilon} > \frac{c^2(2em)^{m-1}3^{k_0-1}m^{3/2}}{\varepsilon} \tag{0.18}$$

since by Equation 0.17

$$k_0 - 1 \leq 4m^3(3m \log m - \log \varepsilon) + \frac{m^{3/2} 2\pi}{\varepsilon}.$$

Step 2 We now show that $y_{k_0-1} > c^2$. We have by Lemma 5.11

$$\begin{aligned}
y_{k_0-1} &\geq 2|a_0|\rho_0^{k_0-1}\frac{\varphi_0}{4} \\
&\geq 2\frac{D}{(2em)^{m-1}}\rho_0^{k_0-1}\frac{\varphi_0}{4} \qquad \text{(by Lemma 5.9)} \\
&\geq \frac{2D}{(2em)^{m-1}}(1/3)^{k_0-1}\frac{\varphi_0}{4} \qquad \text{(since } \rho_0 \geq 1/3) \\
&\geq \frac{D\varepsilon}{(2em)^{m-1}3^{k_0-1}m^{3/2}} \qquad \text{(by Lemma 5.5)} \\
&> c^2 \qquad \text{(by Equation 0.18)}
\end{aligned}$$

as claimed. □

Since $y_{k_0+2} < 0$, the last step of the strategy is Step $k_0 + 1$. If $m \geq 4$, then the sum $\sum_{i=k_0+1-m+2}^{k_0+1} y_i$ includes y_{k_0-1} and, hence, is larger than c. By Lemma 3.6 this implies that there is no strategy to search on m rays in the interval $[1, D]$ with a competitive ratio of c. If $m = 3$ and $\sum_{i=k_0+1-3+2}^{k_0+1} y_i = y_{k_0} + y_{k_0+1} \leq c$, then $y_{k_0-1}/y_{k_0} > c$ and as in Section 4 we see that this also contradicts the existence of a strategy with a competitive ratio of c.

Theorem 5.13 *There is no search strategy for a target on m rays which is contained in the interval $[1, D]$ with a competitive ratio of less than*

$$1 + 2\left(\frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D}\right).$$

Proof: The claim follows directly from Lemma 3.6 since if

$$c \;<\; \frac{m^m}{(m-1)^{m-1}} - \frac{22^2 m^8 \log^2 m}{\log^2 D},$$

then we see as in the case $m = 2$ that Lemma 5.12 implies that there is no strategy Y and integer n such that $y_0 = \cdots = y_{m-1} = D$, all y_i are positive for $0 \leq i \leq n$, $y_{n-m+1}, \ldots, y_n \in [1, c]$, and $\sum_{i=n-m+2}^{n} y_i \leq c$. □

6 An Optimal Strategy

After having proven a lower bound for searching on m rays with an upper bound on the target distance, one of the questions that remains is whether there actually is an optimal strategy that achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ and what it looks like. In this section we present a strategy to search on m rays that achieves the optimal competitive ratio even if the maximum distance D of the target to the starting point is unknown, that is, being told an upper bound on the distance to the target is not a big advantage—even if we consider the convergence rate of the competitive ratio to $1 + 2m^m/(m-1)^{m-1}$ as D increases.

The strategy $X = (x_1, x_2, \ldots)$[1] that achieves a competitive ratio of $1 + 2m^m/(m-1)^{m-1} - O(1/\log^2 D)$ is given by

$$x_i = \sqrt{1 + \frac{i}{m}} \left(\frac{m}{m-1}\right)^i.$$

The competitive ratio of Strategy X in Step $k + m$ is bounded by

$$\begin{aligned}
& 1 + 2\frac{\sum_{j=1}^{k+m-1} \sqrt{1 + \frac{j}{m}} \left(\frac{m}{m-1}\right)^j}{\sqrt{1 + \frac{k}{m}} \left(\frac{m}{m-1}\right)^k} \\
= {} & 1 + 2 \sum_{j=1}^{k+m-1} \sqrt{\frac{m+j}{m+k}} \left(\frac{m}{m-1}\right)^{j-k} \\
= {} & 1 + 2 \left(\sum_{j=1}^{k-1} \sqrt{\frac{j+m}{k+m}} \left(\frac{m}{m-1}\right)^{j-k} + \sum_{j=k}^{k+m-1} \sqrt{\frac{j+m}{k+m}} \left(\frac{m}{m-1}\right)^{j-k} \right) \\
= {} & 1 + 2 \left(\sum_{j=1}^{k-1} \sqrt{\frac{j+m}{k+m}} \left(\frac{m-1}{m}\right)^{k-j} + \sum_{j=0}^{m-1} \sqrt{1 + \frac{j}{k+m}} \left(\frac{m}{m-1}\right)^{j} \right),
\end{aligned}$$

[1] For convenience we start with x_1 instead of x_0.

where we assume for the moment that $k \geq 1$. We present an upper bound for the sums on the right hand side. We first consider the sum

$$\sum_{j=0}^{m-1} \sqrt{1+\frac{j}{k+m}} \left(\frac{m}{m-1}\right)^j.$$

We first observe that

$$\sqrt{1+x} \leq 1+\frac{1}{2}x,$$

for $x \leq 1$, and, therefore,

$$\begin{aligned}
\sum_{j=0}^{m-1} \sqrt{1+\frac{j}{k+m}} \left(\frac{m}{m-1}\right)^j &\leq \sum_{j=0}^{m-1} \left(1+\frac{1}{2}\frac{j}{k+m}\right) \left(\frac{m}{m-1}\right)^j \\
&= \sum_{j=0}^{m-1} \left(\frac{m}{m-1}\right)^j + \frac{1}{2}\sum_{j=0}^{m-1} \frac{j}{k+m} \left(\frac{m}{m-1}\right)^j.
\end{aligned}$$

The first sum is equal to

$$\sum_{j=0}^{m-1} \left(\frac{m}{m-1}\right)^j = \frac{m^m}{(m-1)^{m-1}} - (m-1) \tag{0.19}$$

and the second sum is equal

$$\sum_{j=0}^{m-1} \frac{j}{k+m} \left(\frac{m}{m-1}\right)^j = \frac{(m-1)m}{k+m}. \tag{0.20}$$

Now we consider the sum

$$\begin{aligned}
\sum_{j=1}^{k-1} \sqrt{\frac{j+m}{k+m}} \left(\frac{m-1}{m}\right)^{k-j} &= \sum_{j=1}^{k-1} \sqrt{\frac{k-j+m}{k+m}} \left(\frac{m-1}{m}\right)^j \\
&= \sum_{j=1}^{k-1} \sqrt{1-\frac{j}{k+m}} \left(\frac{m-1}{m}\right)^j.
\end{aligned}$$

Similar to above we observe that

$$\sqrt{1-x} \leq 1-\frac{1}{2}x-\frac{1}{8}x^2,$$

for $x \leq 1$, and, therefore,

$$\sum_{j=1}^{k-1} \sqrt{1-\frac{j}{k+m}} \left(\frac{m-1}{m}\right)^j \leq \sum_{j=1}^{k-1} \left(1-\frac{1}{2}\frac{j}{k+m}-\frac{1}{8}\left(\frac{j}{k+m}\right)^2\right) \left(\frac{m-1}{m}\right)^j.$$

We again compute the values of the sums on the right hand side separately.

$$\sum_{j=1}^{k-1}\left(\frac{m-1}{m}\right)^j = m-1-m\left(\frac{m-1}{m}\right)^k, \tag{0.21}$$

$$\sum_{j=1}^{k-1}\frac{j}{k+m}\left(\frac{m-1}{m}\right)^j = \frac{m(m-1)-(k-m-1)m\left(\frac{m-1}{m}\right)^k}{k+m}, \tag{0.22}$$

and

$$\sum_{j=1}^{k-1}\left(\frac{j}{k+m}\right)^2\left(\frac{m-1}{m}\right)^j = \frac{m(m-1)(2m-1)}{(k+m)^2} - \frac{(k^2+2k(m-2)+2m^2-3m+1)m\left(\frac{m-1}{m}\right)^k}{(k+m)^2}. \tag{0.23}$$

Hence,

$$\sum_{j=1}^{k-1}\sqrt{\frac{j}{k+m}}\left(\frac{m-1}{m}\right)^{k-j} + \sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^j$$
$$\le \sum_{j=1}^{k-1}\left(1-\frac{1}{2}\frac{j}{k+m}-\frac{1}{8}\left(\frac{j}{k+m}\right)\right)\left(\frac{m-1}{m}\right)^j + \sum_{j=0}^{m-1}\left(1+\frac{1}{2}\frac{j}{k+m}\right)\left(\frac{m}{m-1}\right)^j.$$

Equations 0.19 and 0.21 yield

$$\begin{aligned}\sum_{j=0}^{m-1}\left(\frac{m}{m-1}\right)^j + \sum_{j=1}^{k-1}\left(\frac{m-1}{m}\right)^j &= \frac{m^m}{(m-1)^{m-1}} - (m-1) + m - 1 - m\left(\frac{m-1}{m}\right)^k \\ &= \frac{m^m}{(m-1)^{m-1}} - m\left(\frac{m-1}{m}\right)^k. \end{aligned} \tag{0.24}$$

Equations 0.20 and 0.22 yield

$$\begin{aligned}&\frac{1}{2}\sum_{j=0}^{m-1}\frac{j}{k+m}\left(\frac{m}{m-1}\right)^j - \frac{1}{2}\sum_{j=1}^{k-1}\frac{j}{k+m}\left(\frac{m-1}{m}\right)^j \\ &= \frac{1}{2}\frac{(m-1)m}{k+m} - \frac{1}{2}\frac{m(m-1)-(k-m-1)m\left(\frac{m-1}{m}\right)^k}{k+m} \\ &= \frac{1}{2}\frac{(k-m-1)m}{k+m}\left(\frac{m-1}{m}\right)^k. \end{aligned} \tag{0.25}$$

If we combine Equations 0.23, 0.24, and 0.25, then we obtain,

$$\begin{aligned}
&\sum_{j=0}^{m-1}\sqrt{1+\frac{j}{k+m}}\left(\frac{m}{m-1}\right)^{j}+\sum_{j=1}^{k-1}\sqrt{\frac{j+m}{k+m}}\left(\frac{m-1}{m}\right)^{k-j} \\
&\leq \frac{m^m}{(m-1)^{m-1}}-\frac{1}{8}\frac{m(m-1)(2m-1)}{(k+m)^2} \\
&\quad+\left(\frac{1}{2}\frac{k-m-1}{k+m}+\frac{1}{8}\frac{k^2+2k(m-2)+2m^2-3m+1}{(k+m)^2}-1\right)m\left(\frac{m-1}{m}\right)^{k} \\
&\leq \frac{m^m}{(m-1)^{m-1}}-\frac{1}{8}\frac{m(m-1)(2m-1)}{(k+m)^2}
\end{aligned}$$

since

$$\frac{1}{2}\frac{k-m-1}{k+m}+\frac{1}{8}\frac{k^2+2k(m-2)+2m^2-3m+1}{(k+m)^2}\leq 1.$$

There are the two special cases $k=1$ and $k=0$ that remain to be considered. If $k=1$, then we only need to consider

$$\begin{aligned}
1+2\sum_{j=0}^{m-1}\sqrt{1+\frac{j}{1+m}}\left(\frac{m}{m-1}\right)^{j} &\leq 1+2\sum_{j=0}^{m-1}\left(1+\frac{1}{2}\frac{j}{1+m}\right)\left(\frac{m}{m-1}\right)^{j} \\
&\leq 1+2\frac{m^m}{(m-1)^{m-1}}-(m-1).
\end{aligned}$$

If $k=0$, that is the target is detected in the initial m iterations, then the competitive ratio is bounded by

$$1+2\sum_{j=1}^{m-2}\sqrt{1+\frac{j}{m}}\left(\frac{m}{m-1}\right)^{j} \leq 1+2\frac{m^m}{(m-1)^{m-1}}-(m-1).$$

Finally, we relate the number of steps $k+m$ to the distance D to the target. If the target is detected in Step $k+m$, then the distance D to s is in the interval $[\sqrt{1+\frac{k}{m}}(m/(m-1))^k, \sqrt{1+\frac{k+m}{m}}(m/(m-1))^{k+m}]$ and D is bounded from below by

$$\sqrt{1+\frac{k}{m}}\left(\frac{m}{m-1}\right)^{k}\leq D$$

or

$$\frac{1}{2}\log(1+k/m)+k\log\left(1+\frac{1}{m-1}\right)\leq \log D$$

which implies

$$k\leq\frac{\log D}{\log\left(1+\frac{1}{m-1}\right)}\leq (m-1)\log D.$$

Hence,

$$1+2\frac{\sum_{j=1}^{k+m-1}\sqrt{1+\frac{j}{m}\left(\frac{m}{m-1}\right)^j}}{\sqrt{1+\frac{k}{m}\left(\frac{m}{m-1}\right)^k}} \leq 1+2\left(\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{8(\log D + m/(m-1))^2}\right)$$
$$\leq 1+2\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{4\log^2(3D)}.$$

We have shown the following theorem.

Theorem 6.1 *There is a strategy X that achieves a competitive ratio of*

$$1+2\frac{m^m}{(m-1)^{m-1}} - \frac{2m-1}{4\log^2(3D)}$$

if the target is placed at distance $D > 1$ to s.

By Theorem 5.13 the strategy we have presented above is optimal. Note that the lower bound we have shown in Section 5 is only interesting if $\log D > 2m^4 \log m$.

7 Exact Solutions for m=2

To understand the differences between the various searching strategies and bounds presented in this paper, we have charted them for the case $m = 2$ and for distances in the interval $[1, 10000]$.

In figure 5 we plot the best competitive ratio for a distance D. We used the exact optimal strategy derived from the recurrence for searching for a point in two rays at distance of at most D. The x-axis is the distance D plotted in logarithmic scale and the y-axis represents the best competitive ratio attainable for that distance. This curve is contrasted with the lower and upper bounds computed in sections 5 and 6. As you can see, for small values of D the optimal strategy is 5-10% better than the proposed upper bound. Notice as well that the lower bound is quite conservative in this range.

In figure 6 we present the same curves for larger values of D. Notice that while the gap between the lower and upper bound has closed somewhat it is still relatively large. This is due to constant in the second order term being relatively large as compared to the square of the number of digits of $D \approx 10000$.

In figure 7 we compare the competitive ratio attained by the standard doubling strategy, the proposed approximately optimal strategy for unknown D and the exact optimal strategy.

8 Conclusions

We present a lower bound for the problem of searching on m concurrent rays if an upper bound D on the maximal distance to the target is given. We show that in this case the

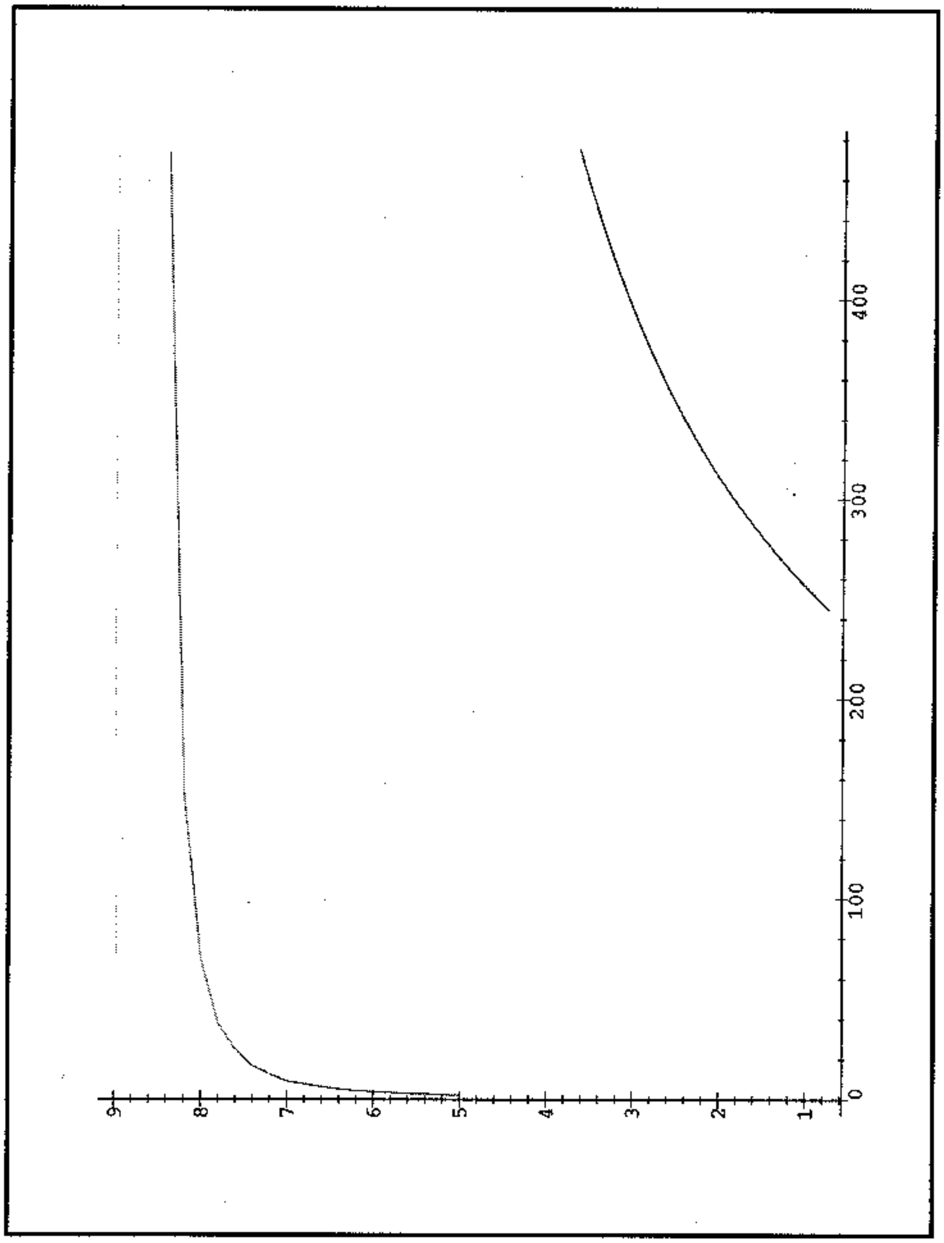

Figure 5: Competitive ratio for optimal strategy $m = 2$, with upper and lower bounds.

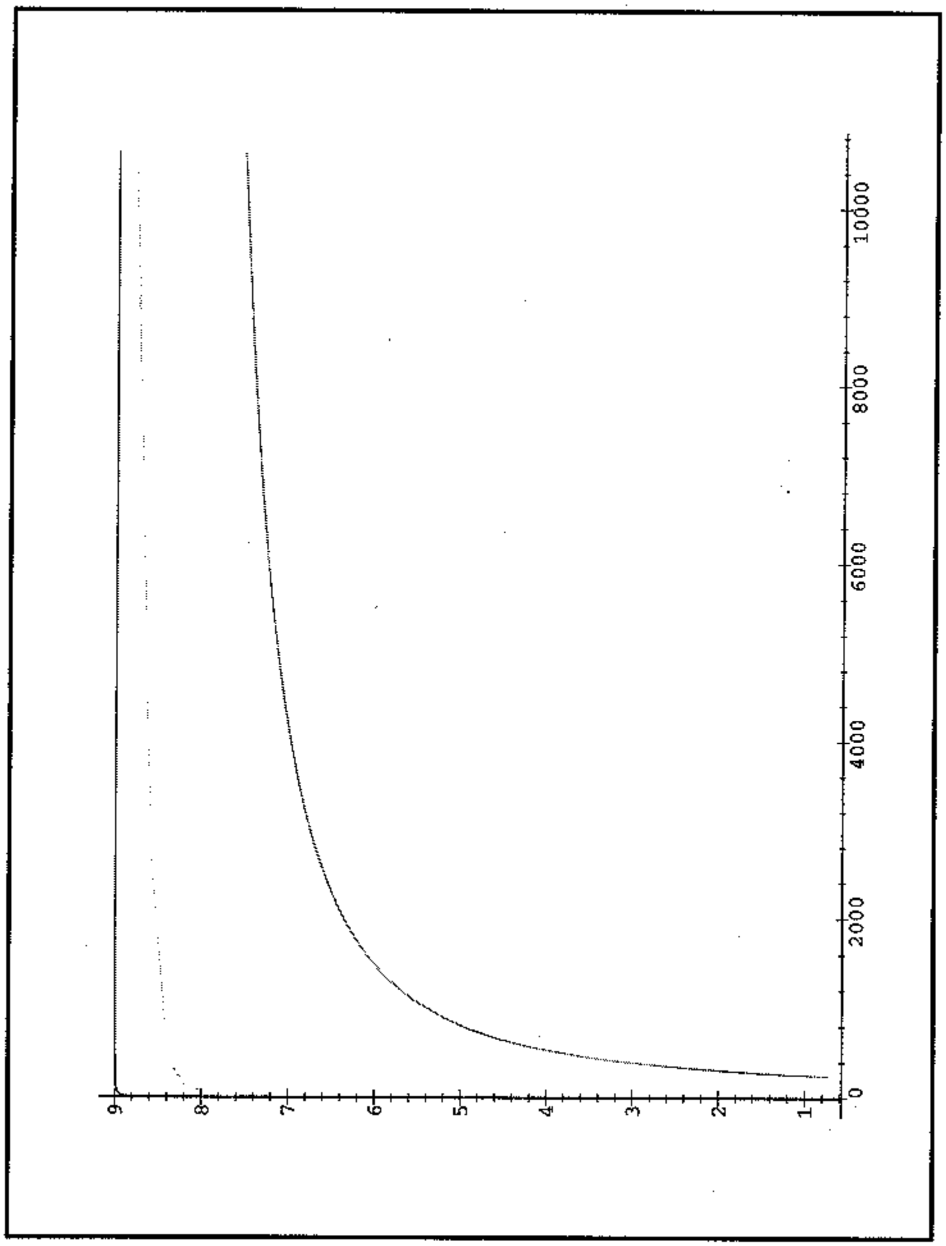

Figure 6: Upper and lower bound and optimal strategy for larger values of D.

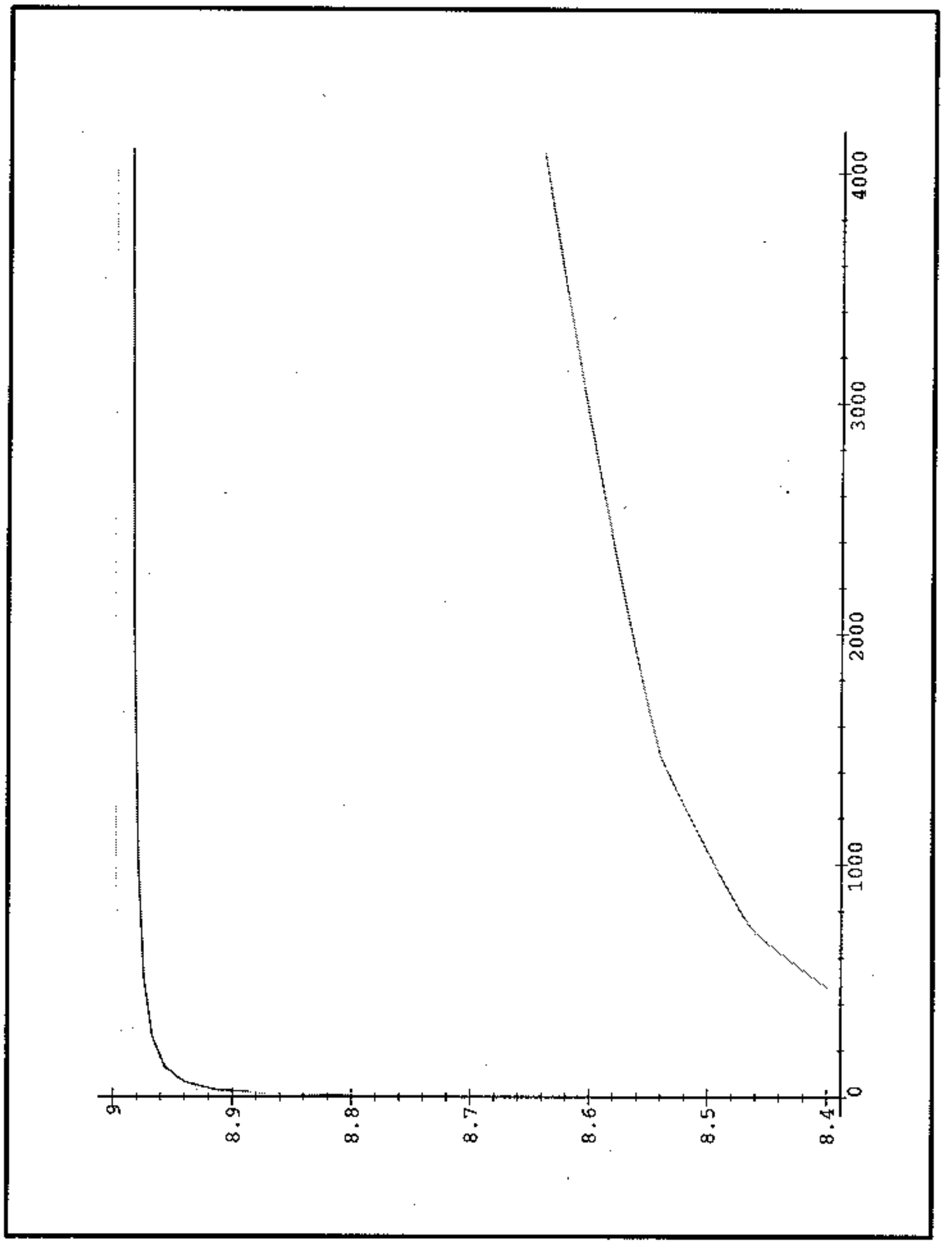

Figure 7: Opimal strategy compared to doubling and upper bound.

competitive ratio of a search strategy is at least $1+2m^m/(m-1)^{m-1}-O(1/\log^2 D)$. Our approach is based on deriving a recursive equation for the step length in each iteration of an optimal strategy. The recursive equation gives rise to a characteristic equation whose roots determine the properties of a strategy. By computing upper and lower bound on the radii and polar angles of the roots we can show that the competitive ratio has to be sufficiently large if the target is far away.

We also present a strategy which achieves a competitive ratio of $1+2m^m/(m-1)^{m-1}-O(1/\log^2 D)$ if the target is detected at distance D. The strategy does not need to know an upper bound on D in advance. Hence, the knowledge of an upper bound on the distance to the target only provides a marginal advantage to the robot—even the convergence rate is not improved.

An interesting open problem is to prove similar results for randomized strategies. One of the problems with randomized strategies is that there is no published proof that there is an optimal periodic strategy. It seems that this is a necessary step before the bounded distance problem can be attacked.

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