Two important results from calculus

- **Taylor's Theorem**

  Theorem 1 ***Let f and its first $n+1$ derivatives be continuous in the closed interval $r \leqslant x \leqslant s$. Then for any x and any a in $r \leqslant x \leqslant s$, we have***

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

where $R_{n+1}(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

and c lies between a and x.

The formula is often written as

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n+1}(x) \tag{1}$$

or, if $h = x - a$,

$$f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + R_{n+1}(h) \tag{2}$$

Important!! You *MUST* know the values of

$f(a), f'(a), \ldots$.

***analytically*, *NOT* just by**

evaluating these functions at a on your calculator!!

Examples of Taylor Series

$f(x) = \cos(x) \quad 0 \leqslant x \leqslant 2\pi$

$f'(x) = -\sin(x),\ \ f''(x) = -\cos(x),$

$f^{(3)}(x) = \sin(x),\ \ f^{(4)}(x) = \cos(x)$

- **Expand** $\cos(x)$ **about** $x = 0$ **(so this can be used for** x **near** 0**.)**

  $f(0) = 1,\ \ f'(0) = 0,\ \ f''(0) = -1,\ \ f^{(3)}(0) = 0,\ \ f^{(4)}(0) = 1$

  $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cos(c)\frac{x^6}{6!}$

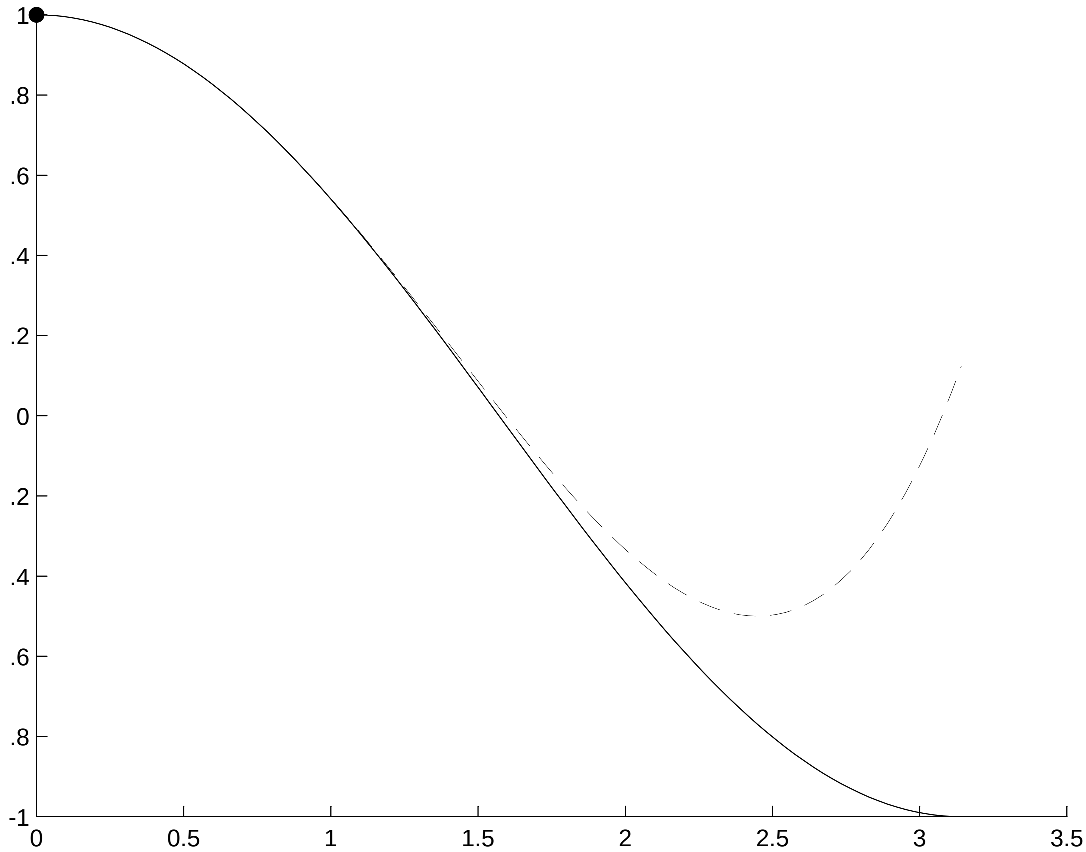

Figure 1: $\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

Taylor Series Cont'd

- **Expand $\cos(x)$ about $x = \frac{\pi}{2}$ (so this can be used for x near $\frac{\pi}{2}$.)**

$$f(\tfrac{\pi}{2}) = 0,\ \ f'(\tfrac{\pi}{2}) = -1,\ \ f''(\tfrac{\pi}{2}) = 0,\ \ f^{(3)}(\tfrac{\pi}{2}) = 1,\ \ f^{(4)}(\tfrac{\pi}{2}) = 0$$

$$\cos(x) = -(x - \tfrac{\pi}{2}) + \tfrac{1}{3!}(x - \tfrac{\pi}{2})^3 - \tfrac{\sin(c)}{5!}(x - \tfrac{\pi}{2})^5$$

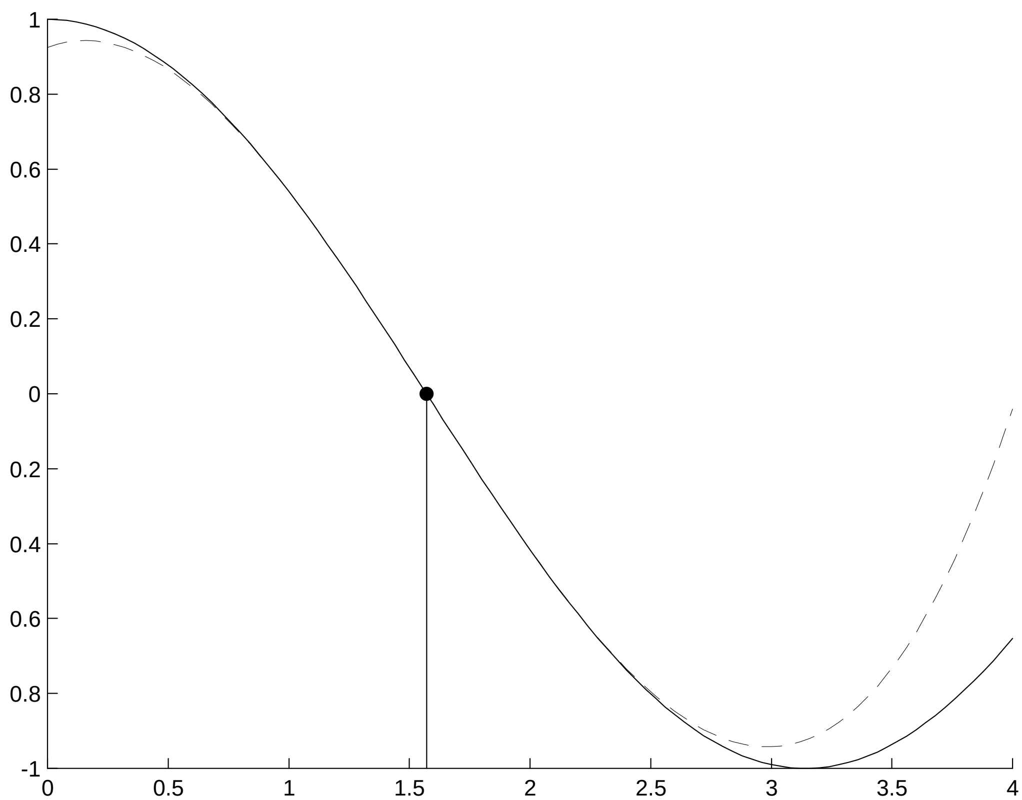

Figure 2: $\cos(x) \approx -(x - \frac{\pi}{2}) + \frac{1}{3!}(x - \frac{\pi}{2})^3$

- **A different view - let $h = x - \frac{\pi}{2}$ (so this can be used for h near 0.)**

$$f(\tfrac{\pi}{2}) = 0,\ \ f'(\tfrac{\pi}{2}) = -1,\ \ f''(\tfrac{\pi}{2}) = 0,\ \ f^{(3)}(\tfrac{\pi}{2}) = 1,\ \ f^{(4)}(\tfrac{\pi}{2}) = 0$$

$$\cos(\tfrac{\pi}{2} + h) = -h + \tfrac{1}{3!}h^3 - \tfrac{\sin(c)}{5!}h^5$$

Taylor Series Application

Compute an approximation to $\sqrt{4.04}$ $(= 2.00997512422418)$. Since we know $\sqrt{4}$, we use the (2) form of the Taylor series with $x = 4.04$, $a = 4$, and $h = x - a = 0.04$. Thus,

$$f(a) = a^{\frac{1}{2}},\ \ f'(a) = \tfrac{1}{2}a^{-\frac{1}{2}},\ \text{ and }\ f''(a) = -\tfrac{1}{4}a^{-\frac{3}{2}}$$

$$f(4) = 4^{\frac{1}{2}} = 2,\ \ f'(4) = \tfrac{1}{2}4^{-\frac{1}{2}} = \tfrac{1}{4},\ \text{ and }\ f''(4) = -\tfrac{1}{4}4^{-\frac{3}{2}} = -\tfrac{1}{32}$$

$$\begin{aligned} f(a+h) &\approx f(a) + f'(a)h \\ &= 2 + \tfrac{1}{4}(0.04) = 2.01 \\ &\quad \text{rel err} = \left|\tfrac{2.01-\sqrt{4.04}}{\sqrt{4.04}}\right| \leqslant 0.125 \times 10^{-4} \\ f(a+h) &\approx f(a) + f'(a)h + \tfrac{f''(a)}{2}h^2 \\ &= 2.01 - \tfrac{1}{2}\cdot\tfrac{1}{32}(0.04)^2 \\ &= 2.01 - 0.000025 = 2.009975 \\ &\quad \text{rel err} \leqslant 6.1804 \times 10^{-8} \end{aligned}$$

Note that the first neglected term in deriving the linear approximation is

$$\frac{1}{2}\cdot\frac{1}{32}(0.04)^2 = 0.25 \times 10^{-4}$$

which *approximates* the absolute error for the the linear approximation.

Mean Value Theorem

Theorem 2 ***Let f be continuous in the closed interval $a \leqslant x \leqslant b$ and differentiable in the open interval $a < x < b$. Then there is a point c in $a < x < b$ at which***

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Can be rewritten as**

$$f(b) = f(a) + f'(c)(b - a) \qquad a < c < b$$

Mean Value Theorem - Example

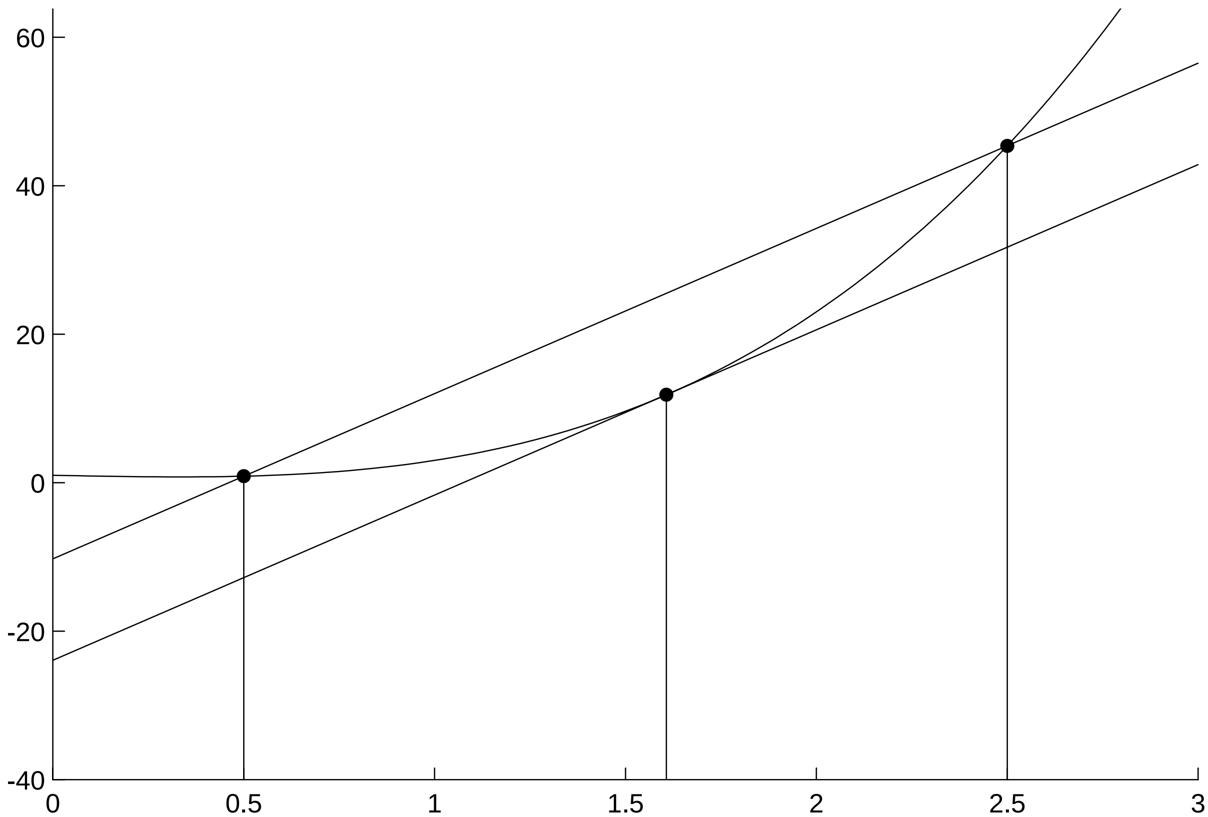

Figure 3: **Mean Value Theorem**

Consider the polynomial

$$p(x) = 1 - x + 3x^3 \qquad 0.5 \leqslant x \leqslant 2.5$$

Then $p(x)$ **goes through the points** $(0.5,\ 0.875)$ **and** $(2.5,\ 45.375)$**.**

The *line* through these two points is given by

$$l_1(x) = -10.25 + 22.25x, \qquad \text{slope} = 22.25$$

Now

$$p'(c) = -1 + 9c^2 = 22.25 \Longrightarrow c = 1.6073 \quad \text{and} \quad p(c) = 11.8491$$

The tangent line to $p(x)$ **at** $x = 1.6073$ **is given by**

$$l_2(x) = -23.9128 + 22.25x, \qquad \text{slope} = 22.25$$