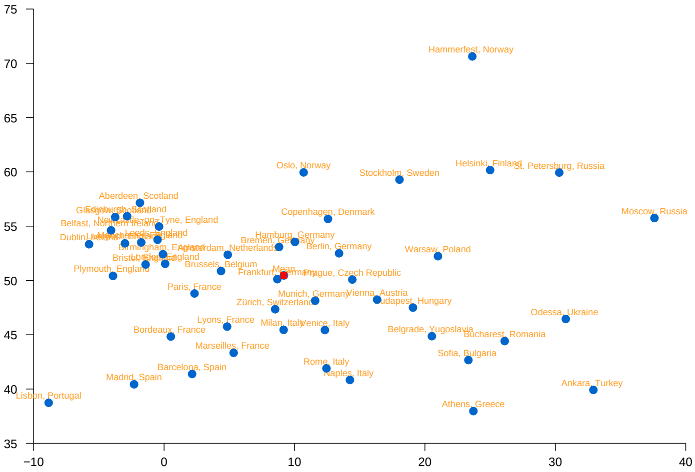
Primal–Dual Algorithms for Halfspace Depth

David Bremner   Komei Fukuda   Vera Rosta

March 17, 2006

Wir sind Zentrum



Estimators of Location

centre Given a set of vectors, return a vector which "best describes" the set.

City	halfplane depth-1
Frankfurt, Germany	19
Brussels, Belgium	17
Munich, Germany	16
Amsterdam, Netherlands	15
Zürich, Switzerland	13
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Prague, Czech Republic	12

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depth measure Rank a set of vectors such that vectors of maximum rank define one or more centre vectors.

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- The breakdown point of the mean is $\frac{1}{n}$ (i.e. one error suffices to destroy the estimate).
- The median in $\mathbb{R}^1$ has breakdown $1/2$.

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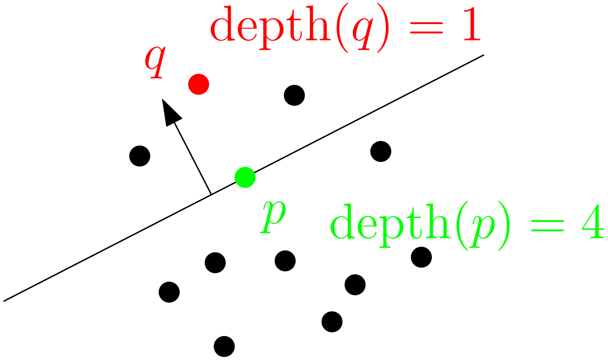
Nesting Let $D_{X,k}$ denote the points of $\mathbb{R}^d$ at depth k with respect to X. We want

$$k > j \implies D_{X,k} \subseteq \operatorname{conv} D_{X,j}$$

Halfspace Depth

The *halfspace depth* of a point q with respect to $S \subset \mathbb{R}^d$ is defined as

$$\mathrm{depth}_S(q) = \min_{a \in \mathbb{R}^d \setminus 0} |\{p \in S \mid \langle a, p \rangle \geq \langle a, q \rangle\}|$$



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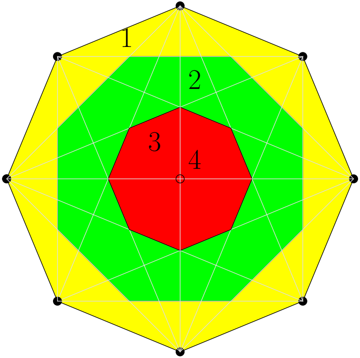
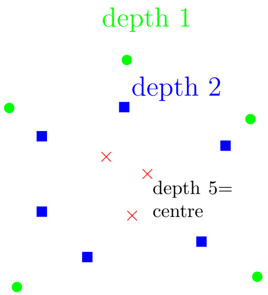


Tukey Median

The *Tukey Median* $t(S)$ is defined as

$$\{q \in S \mid \mathrm{depth}_S(q) = \max_{p \in S} \mathrm{depth}_S(p)\}$$



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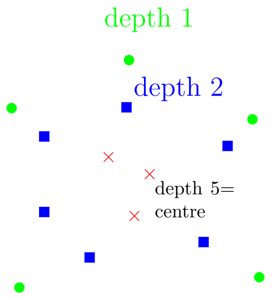

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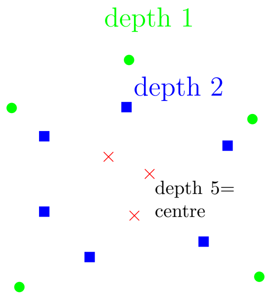

- Halfspace depth is nested
- The Tukey median has breakdown point at least $1/(d+1)$ for points in general position.

Complexity results

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  - Densest Open and Closed Hemisphere problem
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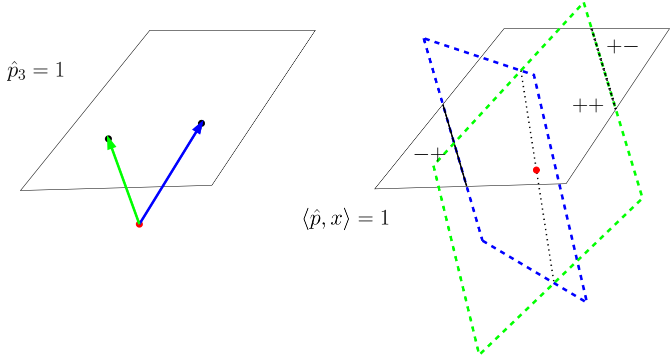
  - Maximum Feasible Subsystem
  - A 2-approximation of MFS is possible.

The Dual Arrangement



$$\hat{p} = (p, 1) \in \mathbb{R}^{d+1}$$

$$h(p) = \{x \in \mathbb{R}^{d+1} \mid \langle \hat{p}^i, x \rangle = 0\}$$

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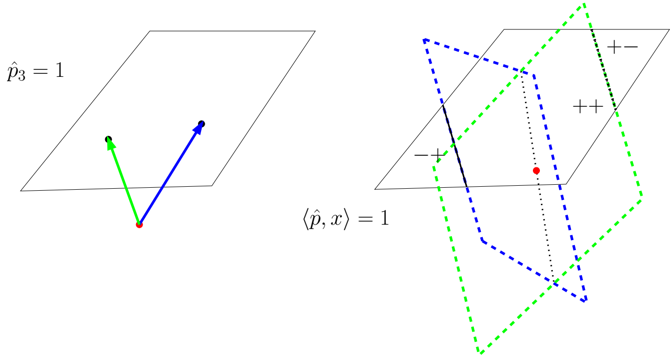

$$A_p = \{h(q) \mid q \in S \setminus \{p\}\} \cap \{x \mid \langle \hat{p}, x \rangle = 1\}$$

$$\sigma(x) = (\sigma_1 \ldots \sigma_n)$$

$$\text{where} \quad \sigma_i = \text{sign}(\langle \hat{p}, x \rangle)$$

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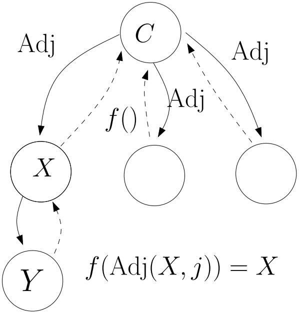
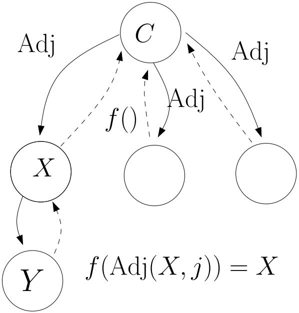


Reverse Search

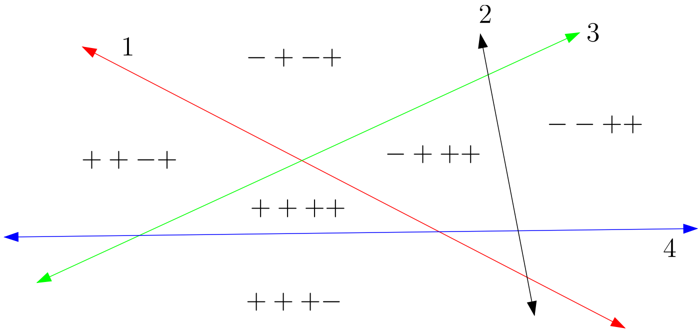
- *reverse search* requires two problem specific functions.
  - The *adjacency oracle* Adj() returns the neighbouring cells
  - The *local search function* $f(\cdot)$ satisfies

$$\exists C\ \forall X\ \exists k\ f^k(X) = C$$



Adjacency Oracle

- $X \equiv \sigma(x)$ is *flippable* at position j if negating sign j yields a cell.



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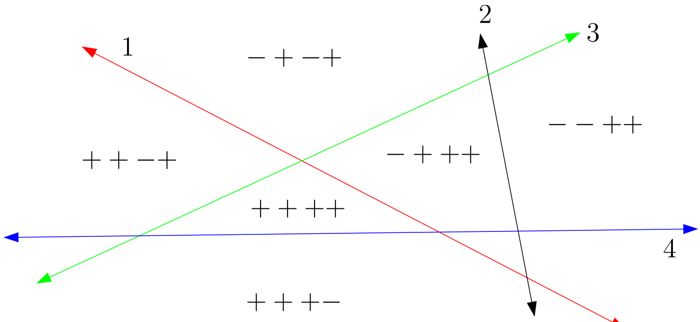
Adjacency Oracle

- $X \equiv \sigma(x)$ is *flippable* at position j if negating sign j yields a cell.
- $\mathrm{Adj}(X, j)$ is true iff X is flippable at position j. Solve via LP.



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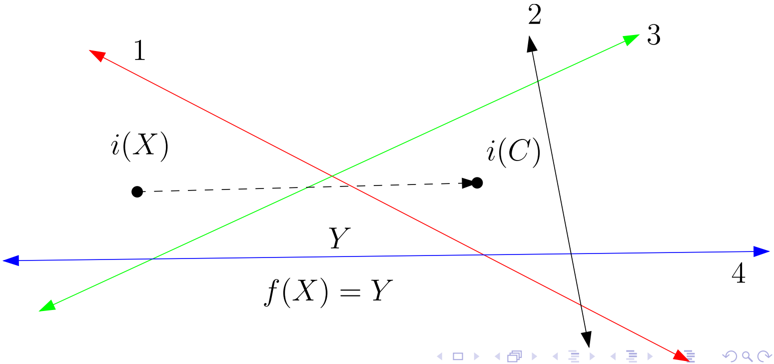
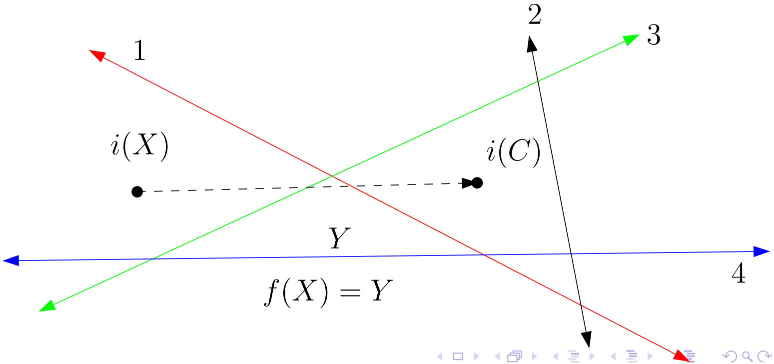
Local Search Function

- Define a canonical interior point $i(X)$ for each cell.



Local Search Function

- Define a canonical interior point $i(X)$ for each cell.
- Choose an arbitrary cell C.
- To find a closer cell to C "shoot a ray" from $i(X)$ to $i(C)$.



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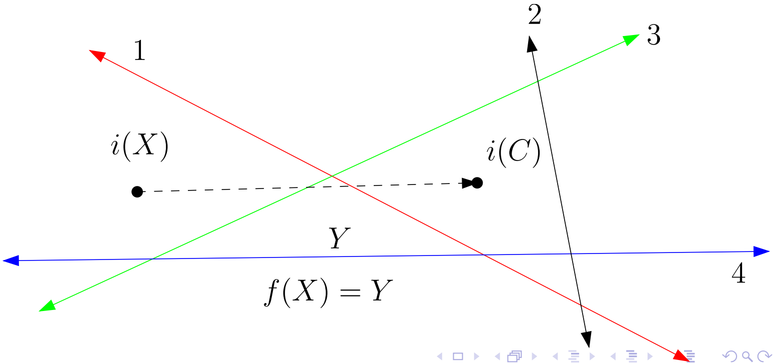
- To find a closer cell to C "shoot a ray" from $i(X)$ to $i(C)$.
- Requires a single LP for $i(C)$. $i(X)$ can be computed by flipping test.



Reverse Search Summary

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- Little information until enumeration terminates.

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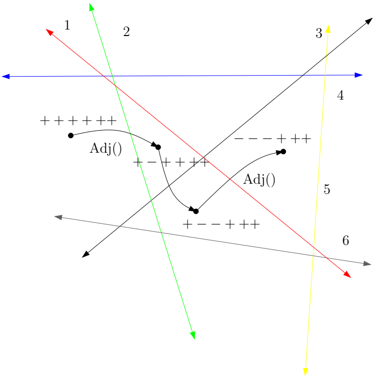
- To ensure termination, fall back on enumeration after a fixed time limit.
- Generally, answers improve with time.

Upper Bounds via Random Walks

- Use Adj() oracle from enumeration algorithm



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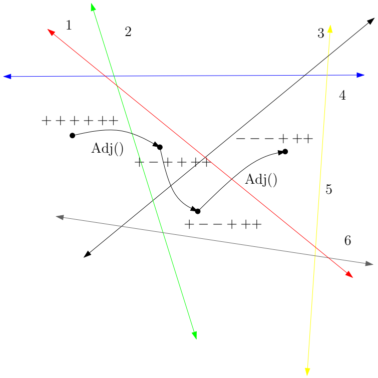

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- Use Adj() oracle from enumeration algorithm
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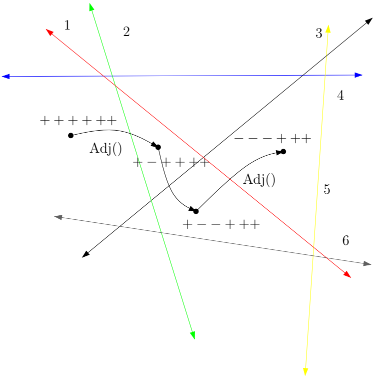
- Repeat several times choosing a random starting cell.



Minimal Dominating Sets

Definition

A *Minimal Dominating Set* (MDS) for $p \in \mathbb{R}^d$ with respect to $S \subset \mathbb{R}^d$ is $R \subseteq S$ such that

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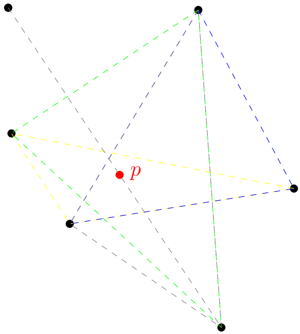

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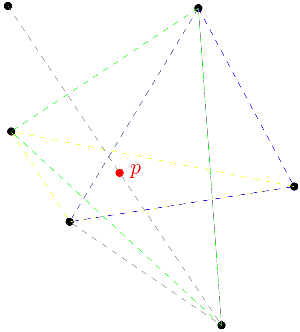

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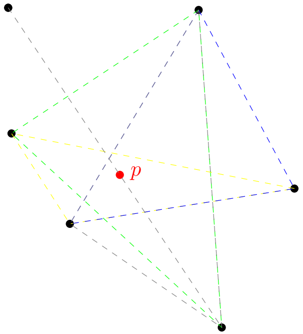
- $p \in \operatorname{conv} R$
- if $R' \subsetneq R$ then $p \notin \operatorname{conv} R'$.

An MDS might also be called a Charathéodory set.



Lower bounds via MDS's

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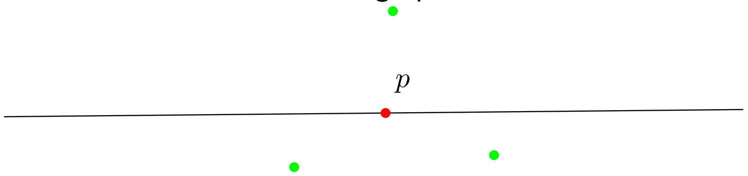
Each MDS intersects both closed sides of any hyperplane through p.



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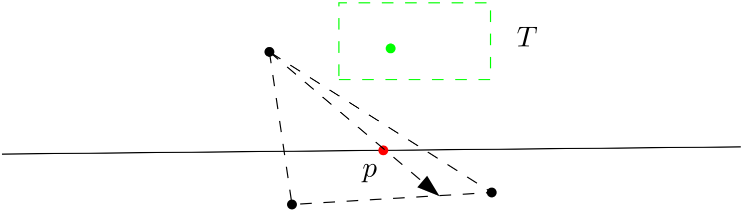

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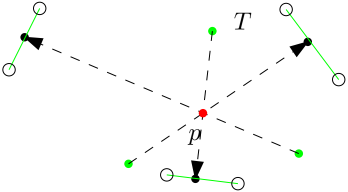

Generating Missed MDSs (cuts)

Definition

Given a partial traversal T for the MDS's of p w.r.t. S Define $\bar{S} = S \setminus T$. Define the *auxiliary polytope* $Q(p, T)$ as λ satisfying:

$$\lambda \bar{S} = p$$

$$\sum_i \lambda_i = 1 \qquad \lambda_i \geq 0$$

- Each vertex (basic solution) of $Q(p, T)$ defines an MDS missed by T.

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- k cuts can be found via reverse search (or other pivoting method).

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4. Otherwise solve the resulting (partial) transversal problem via integer programming.

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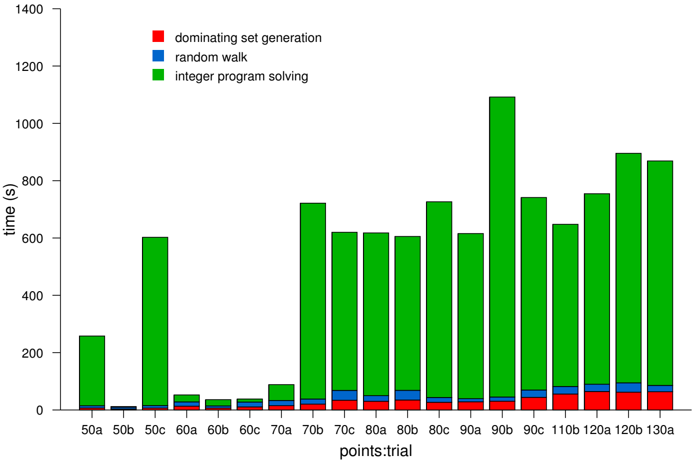
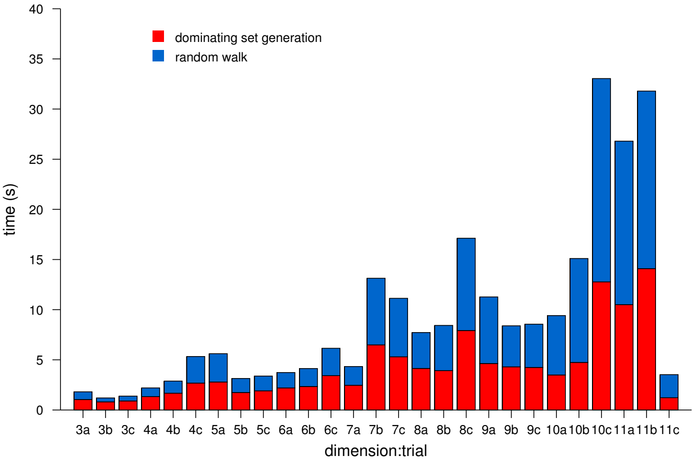
5. If bored, switch to enumeration.

Software

Software	Where	Purpose
cdd	http://www.ifor.math.ethz.ch/~fukuda/cdd_home	Solving Dense LPs
COIN	http://www.coin-or.org	Simplex Solver, cut generation
lrs	http://cgm.cs.mcgill.ca/~avis/	MDS generation.
SYMPHONY	http://www.branchandcut.org	Branch and Cut
ZRAM	http://www.cs.unb.ca/~bremner/zram	Parallel reverse search

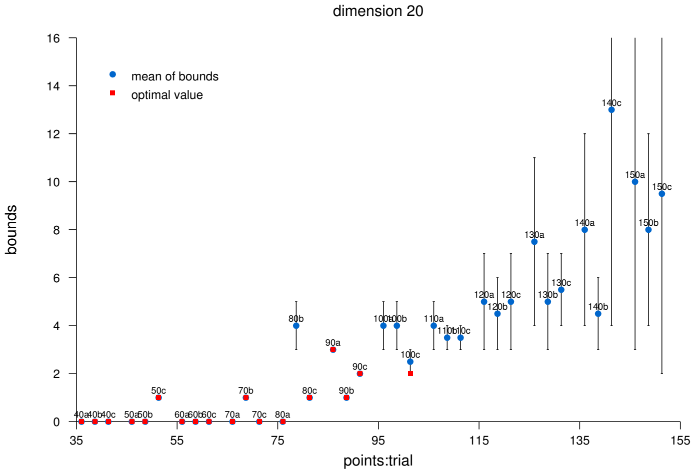


Dimension 10

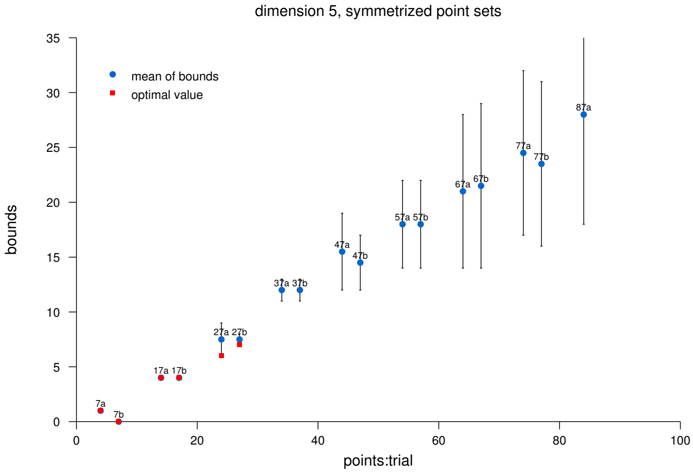


dominating set generation
random walk
time (s)
dimension:trial
0
5
10
15
20
25
30
35
40
3a 3b 3c 4a 4b 4c 5a 5b 5c 6a 6b 6c 7a 7b 7c 8a 8b 8c 9a 9b 9c 10a 10b 10c 11a 11b 11c

50 points



dimension 20
mean of bounds
optimal value
bounds
points:trial


dimension 5, symmetrized point sets
mean of bounds
optimal value
bounds
points:trial
7a
7b
17a 17b
27a 27b
37a 37b
47a
47b
57a 57b
67a 67b
77a
77b
87a
0
5
10
15
20
25
30
35
0
20
40
60
80
100

Conclusions

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- Better upper bounds would nice.