

Facet Generation and Symmetric Triangulation

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June 19, 2008

with Achill Schürmann, Mathieu Dutour Sikirić

Facet enumeration up to symmetry

Definition

Linear transformation A is a *restricted automorphism* for $\text{cone}(V)$ if

$$\{Av \mid v \in V\} = V$$

$\overline{\text{Aut}}(V)$ denotes the group of restricted automorphisms of $\text{cone}(V)$.

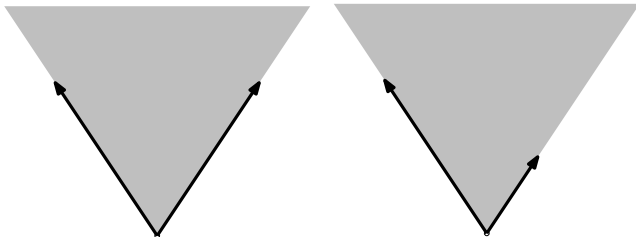
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Problem

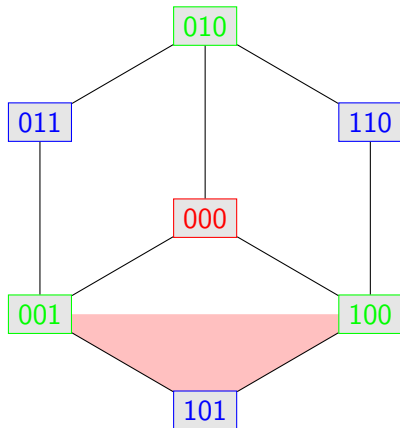
Given $V \subseteq \mathbb{R}^d$, $\overline{\text{Aut}}(V)$.

Find One representative of each orbit of facet defining inequalities for $\text{cone}(V)$.

Bases and Orbits

basis $(r - 1)$ rays (d vertices)
spanning a facet.

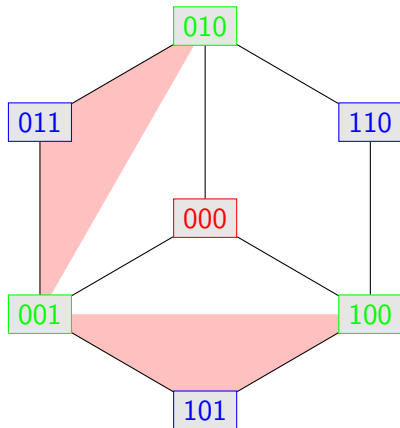
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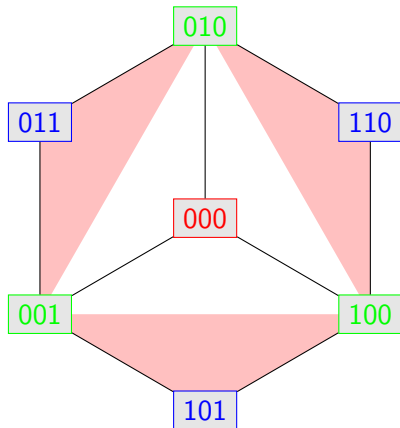
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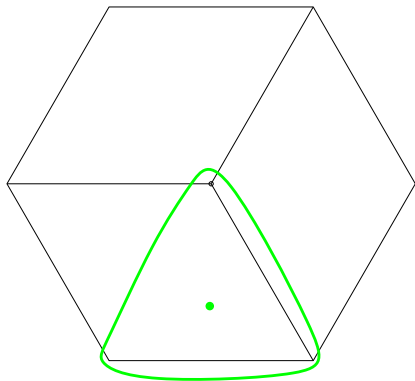
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Exploring the Basis Graph

pivot $C' = C \setminus \{l\} \cup \{e\}$
such that C' is a basis.

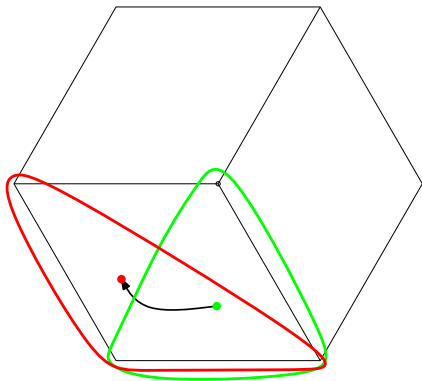
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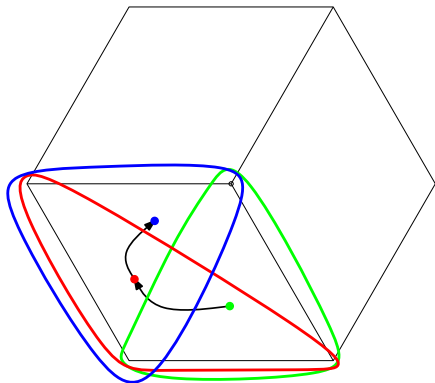
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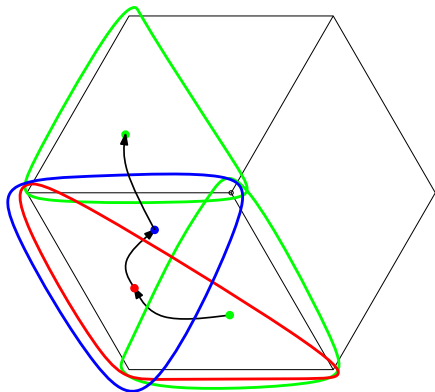
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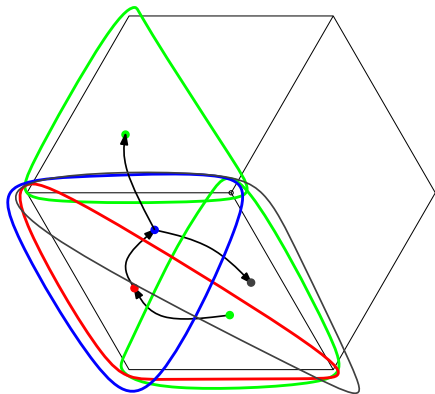
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Wreath products

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Let $P = \text{conv}(v_1 \dots v_m) \subset \mathbb{R}^d$. Let

$Q = \text{conv}(w_1 \dots w_n) \subset \mathbb{R}^e$.

$$P \wr Q = \text{conv} \begin{bmatrix} P & 0 & 0 & & 0 \\ 0 & P & 0 & & 0 \\ 0 & 0 & P & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & P \\ w_1 & w_2 & w_3 & & w_n \end{bmatrix}$$

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Roughly, $\overline{\text{Aut}}(Q)$ acts on “big columns” and $\overline{\text{Aut}}(P)$ *within* them.

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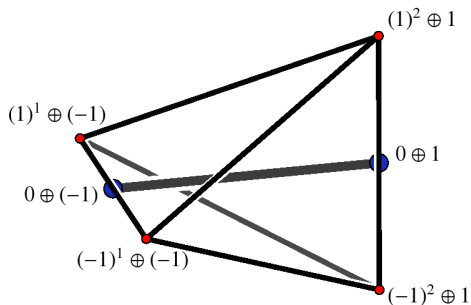
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$$[-1, 1] \wr [-1, 1]$$

(courtesy of Joswig and Lutz (2005))

Wreath Products of Cross Polytopes

Example

Let $C_k = \text{conv}\{\pm e_1, \dots, \pm e_k\}$. Let $P = C_d \wr C_e$.

- ▶ P has dimension $D = 2de + e$ and $4de \sim 2D$ vertices
- ▶ P has $2^{(d+1)e}$ facets, each containing $3de \sim 1.5D$ vertices
- ▶ P has one orbit of vertices, facets, and $(D - 1)$ -bases.

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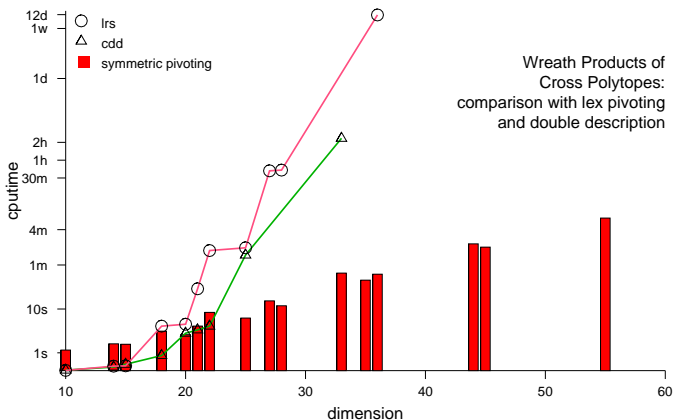
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Orbitwise Degenerate Polytopes

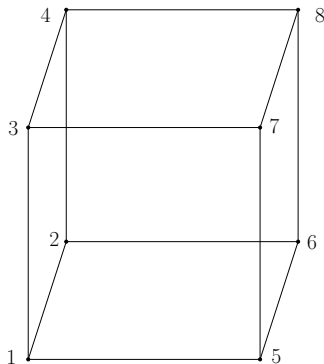
	Dimension	Triangulation Δ s	Basis Orbits
Cut	10 ($n = 5$)	496	2
	15 ($n = 6$)	186636	6300
Cubes	4	48	4
	5	240	17
	6	1440	237
	7	10080	9892
	8	80640	> 209000

Valid Perturbation

Definition

\tilde{V} is a *valid perturbation* of V if $\exists \nu(\cdot) : V \leftrightarrow \tilde{V}$ such that $\forall W \subseteq V$,

1. If $\nu(W)$ is linearly dependent then W is.
2. If $\nu(W)$ is extreme for \tilde{V} then W is extreme for V .

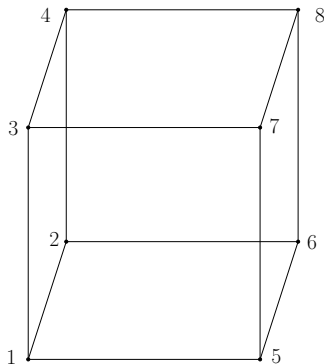


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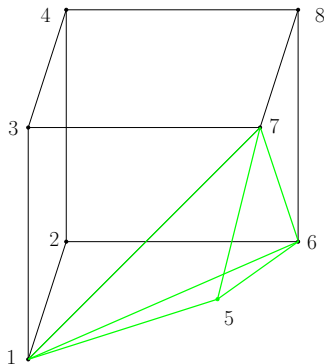


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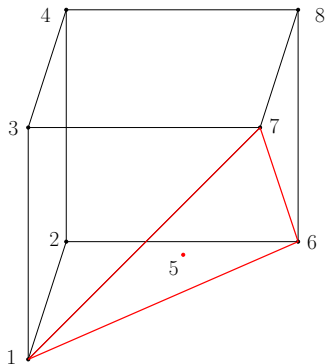


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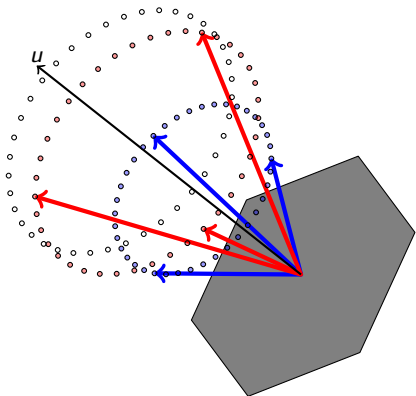
Symmetry preserving perturbation

Proposition

- ▶ Let $V \subset \mathbb{R}^d$. V_1, \dots, V_k the orbits of V under H , and u be a fixed point for H ,
- ▶ There exists $\varepsilon_1 \gg \dots \gg \varepsilon_k \geq 0$ such that

$$V' = \bigcup_j (V_j \pm \varepsilon_j u)$$

is a valid perturbation of V and $H \leq \overline{\text{Aut}}(V')$.



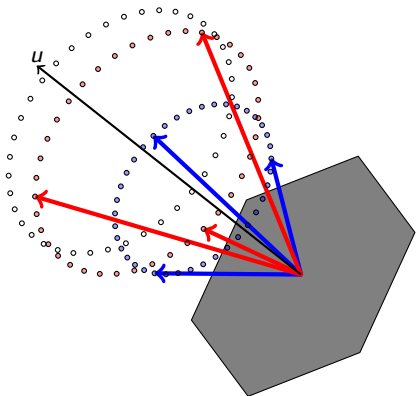
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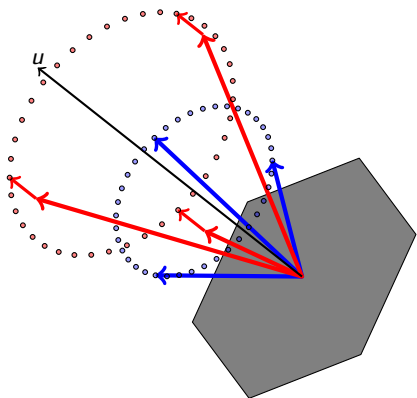
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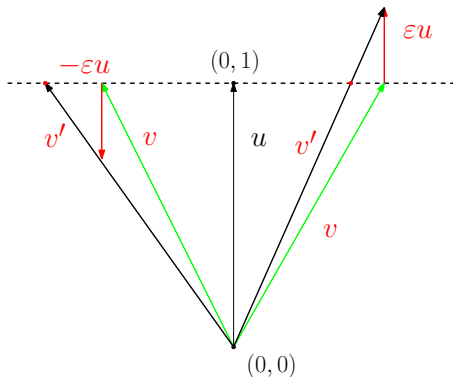
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Affine Orbitwise Perturbation

Perturbation by Scaling

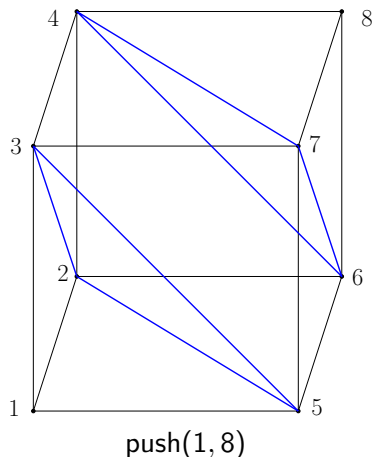
Linear	Affine	Name
$(p, 1) + \varepsilon(0, \dots, 0, 1)$	$\frac{p}{1 + \varepsilon}$	push
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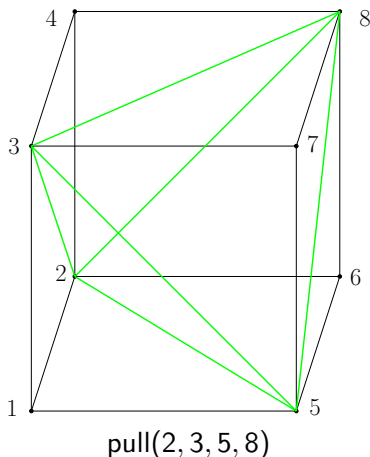
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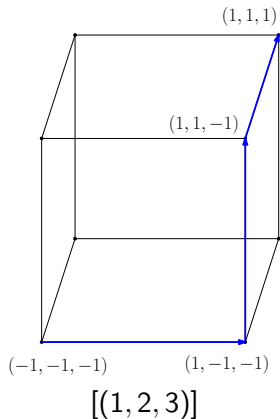
Linear Ordering Triangulation

Definition

- ▶ Let $I^d = [-1, 1]^d$. Let $\mathbf{e} = (1, \dots, 1)$.
- ▶ For each $\rho \in \text{Sym}(d)$, there is a path $[\rho]$ from $-\mathbf{e}$ to \mathbf{e} .
- ▶ Define Δ_ρ as $\text{conv}[\rho]$.
- ▶ The *linear ordering triangulation* of $\text{bdy } I^d$ is the intersection of $\text{bdy } I^d$ with all Δ_ρ .

Proposition

$H_d = \text{stab}(\overline{\text{Aut}(I^d)}, \{-\mathbf{e}, \mathbf{e}\})$ acts transitively on the l.o.t. of $\text{bdy } I^d$.



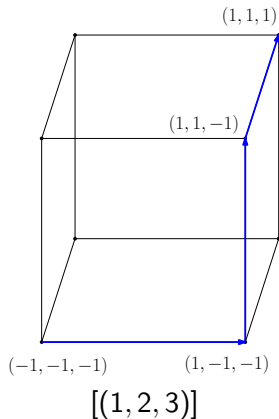
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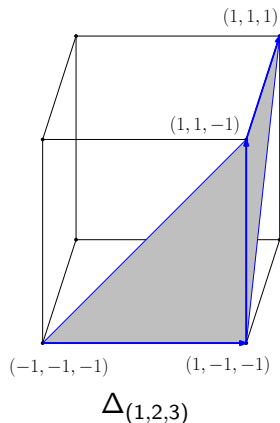
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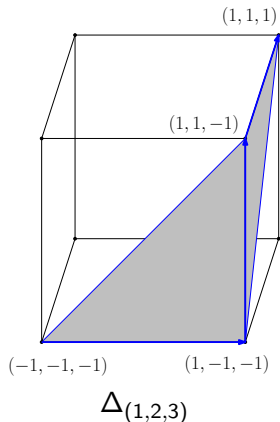
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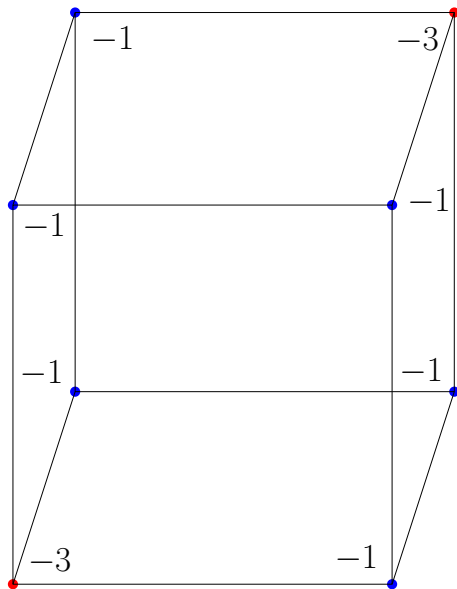
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Linear Ordering Perturbation

Example

Let \tilde{I}^d denote the H_d -orbitwise pulling of I^d in order induced by $\omega(v) = \min(\mathbf{e}^T v, -\mathbf{e}^T v)$.

- ▶ $\text{bdy } \tilde{I}^d \equiv$ the l.o.t. of $\text{bdy } I^d$.
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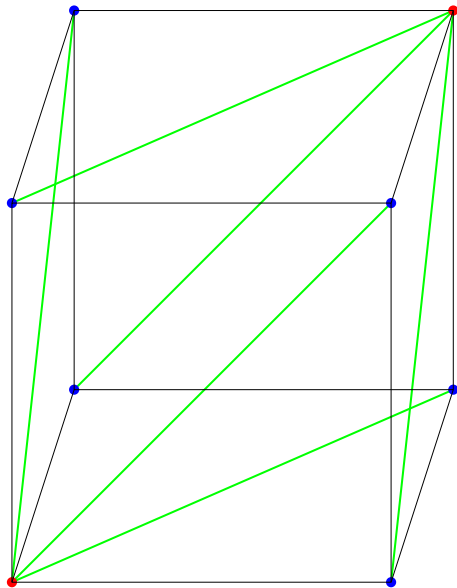


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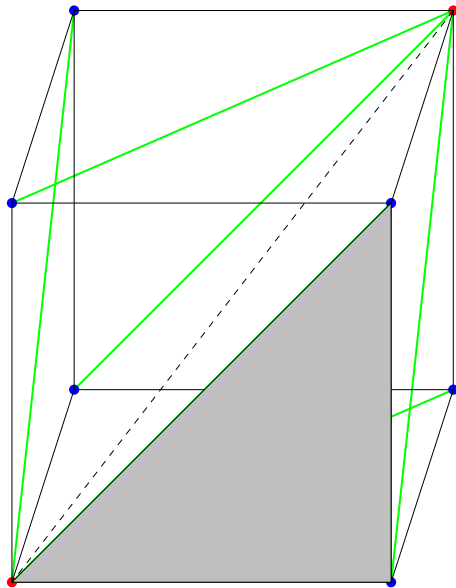


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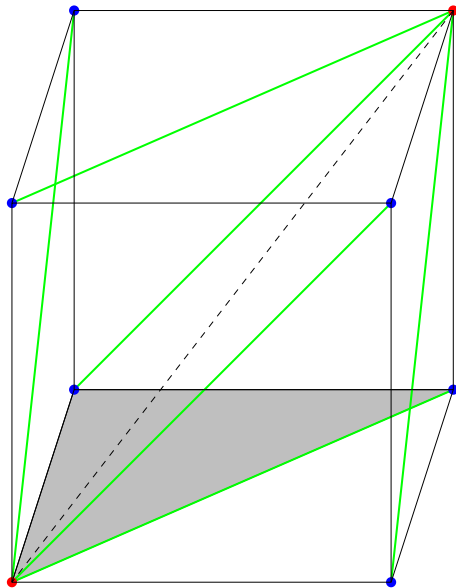


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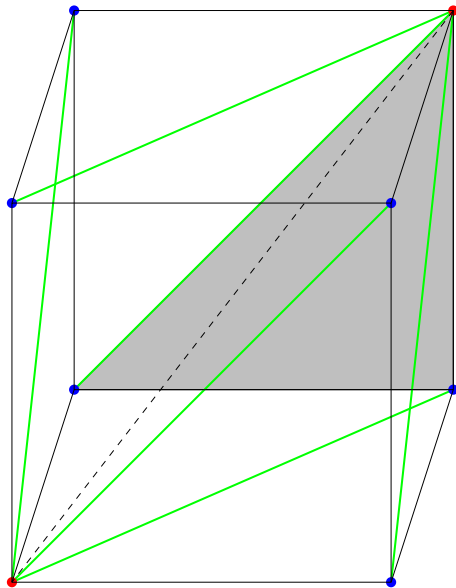


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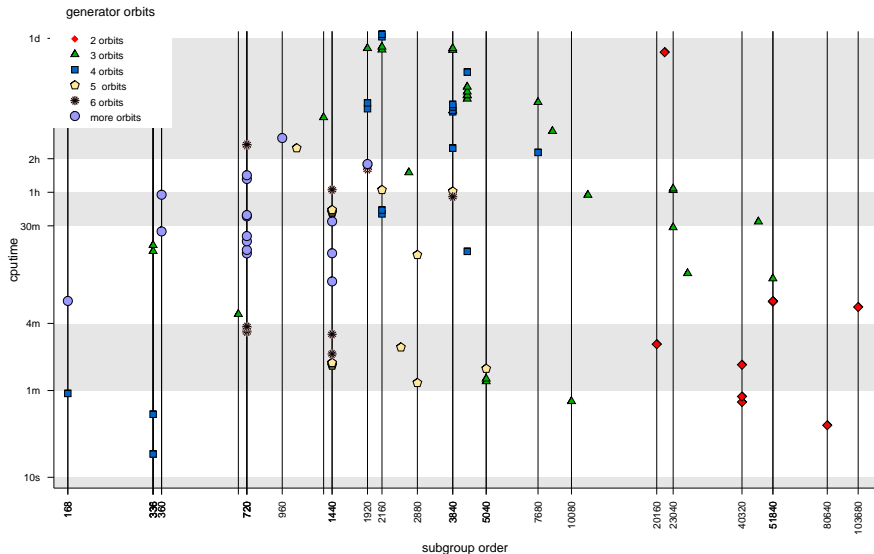


Example: E_7 root lattice contact polytope

Contact Polytope for E_7 root lattice

		Orbits
Dimension	8	
Group Order	2903040	
Vertices	126	1
Facets	632	2
Irs Δ 's	20520	
bases		161

What makes a good subgroup?



Conclusions

- ▶ For certain special cases, pivoting works well for facet generation under symmetry.
- ▶ The question of what polytopes have symmetric triangulations is an interesting one.
- ▶ Simple heuristics exist to find subgroups with desired size and number of input orbits; more ideas are probably needed to find effective triangulations.

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