# CS3383 Unit 1: Divide and Conquer Introduction

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January 7, 2018

# unit prereqs

- mergesort
- geometric series (CLRS A.5)

# Contents

Divide and Conquer

- Intro
- Merge Sort
- Recursion Tree for recurrences
- Integer Multiplication

# Structure of divide and conquer

```
function SOLVE(P)
    if |P| is small then
        SolveDirectly(P)
    else
        P_1 ... P_k = Partition(P)
        for i = 1 ... k do
            S_i = Solve(P_i)
        end for
        Combine(S_1 ... S_k)
    end if
end function
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- Where is the actual work?

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- How many subproblems?
- How big are the subproblems?

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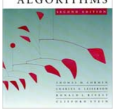
# Merge Sort

Figure 2_4 in DPV

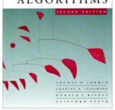


# Merge sort

MERGE-SORT $A[1 \mathinner{\ldotp\ldotp} n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \mathinner{\ldotp\ldotp} \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \mathinner{\ldotp\ldotp} n\,]$ .
3. "***Merge***" the 2 sorted lists.

***Key subroutine:*** MERGE



# Merging two sorted arrays

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|---|---|
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| 13 | 11 |
| 7 | 9 |
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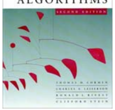


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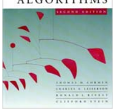


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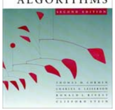

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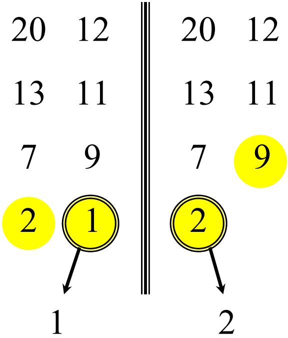

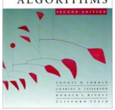

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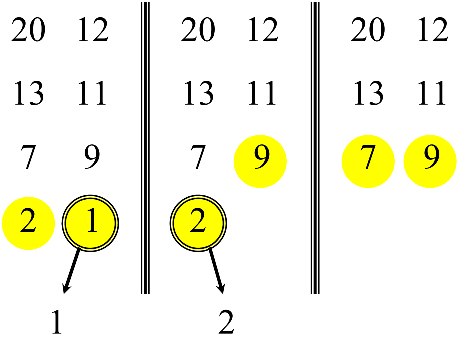

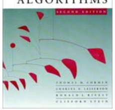

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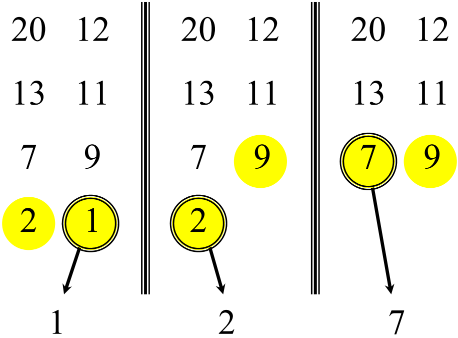

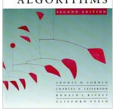

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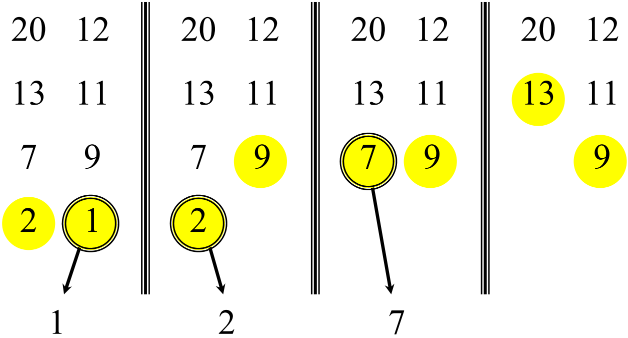

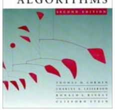

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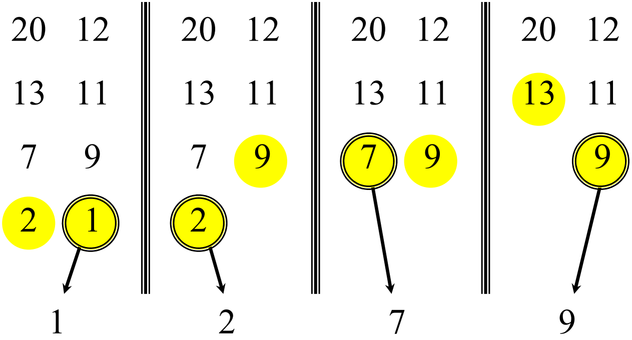

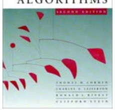

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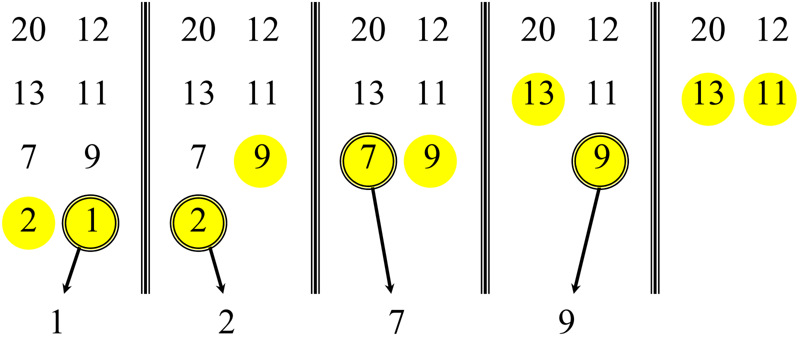

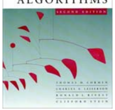

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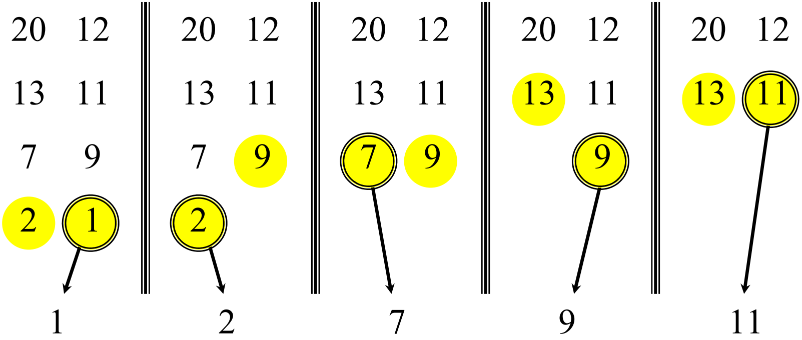

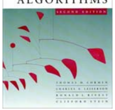

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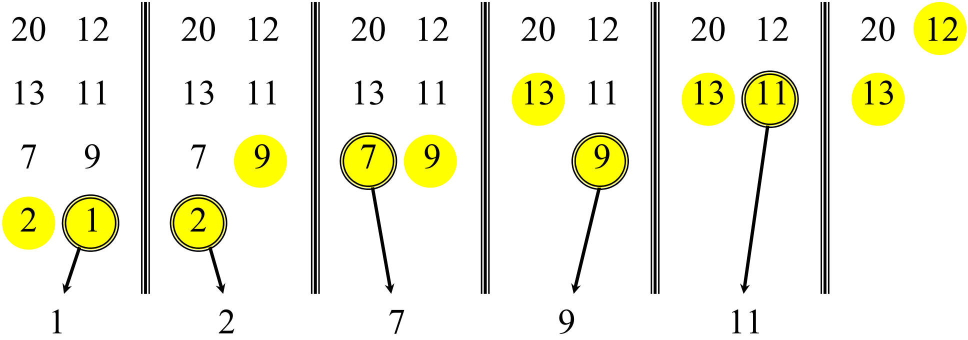

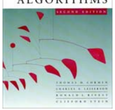

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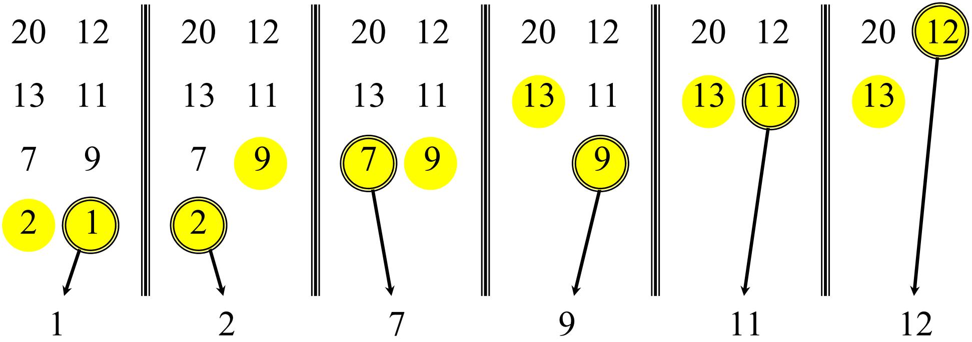

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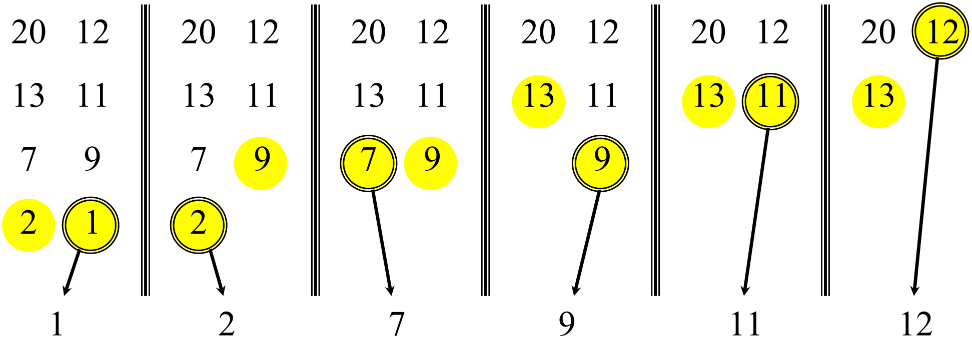
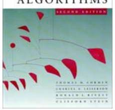


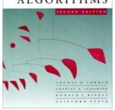

Time = $\Theta(n)$ to merge a total of $n$ elements (linear time).



# Analyzing merge sort

| | |
|---|---|
| $T(n)$ | **MERGE-SORT** $A[1\,..\,n]$ |
| $\Theta(1)$ | 1. If $n = 1$, done. |
| $2T(n/2)$ | 2. Recursively sort $A[\,1\,..\,\lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\,..\,n\,]$. |
| $\Theta(n)$ | 3. ***"Merge"*** the 2 sorted lists |

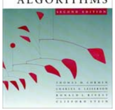
***Sloppiness:*** Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.



# Recurrence for merge sort

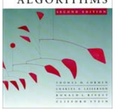
$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small $n$, but only when it has no effect on the asymptotic solution to the recurrence.
- We will see several ways starting with "Rec. Tree" to find a good upper bound on $T(n)$.



# Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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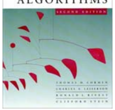


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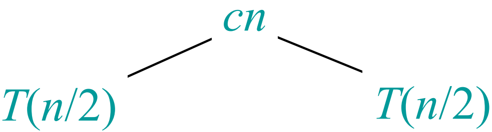

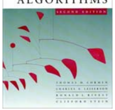

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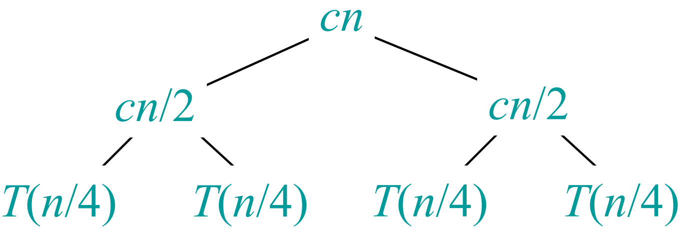

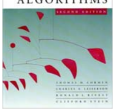

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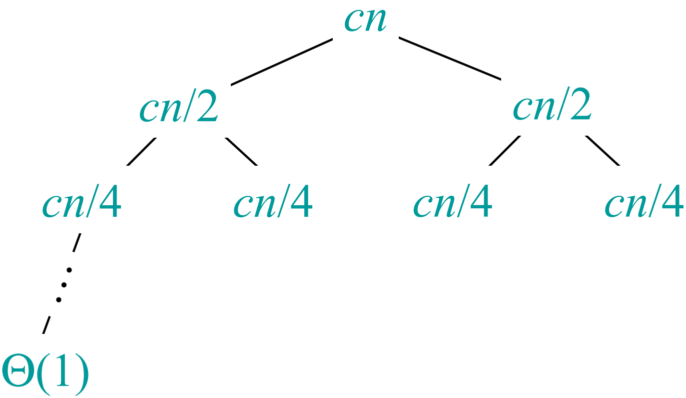

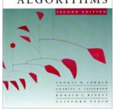

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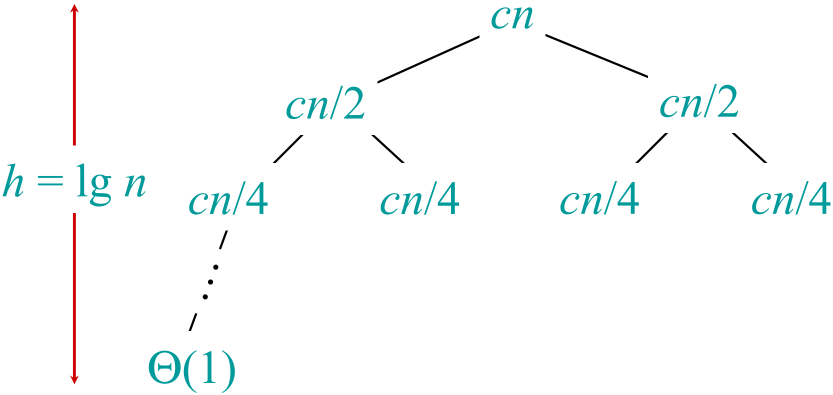

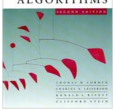

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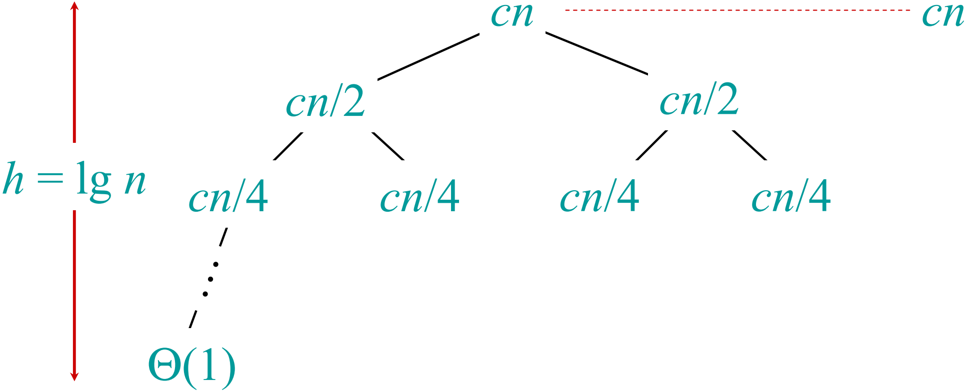

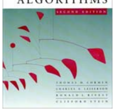

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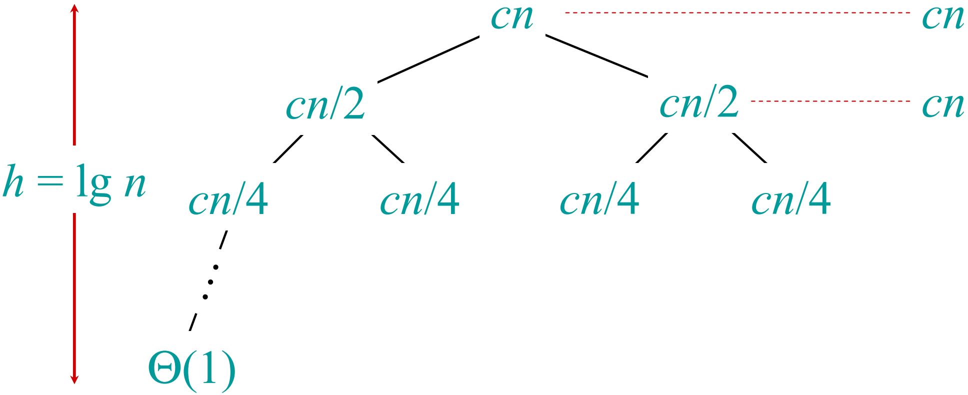

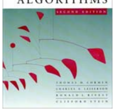

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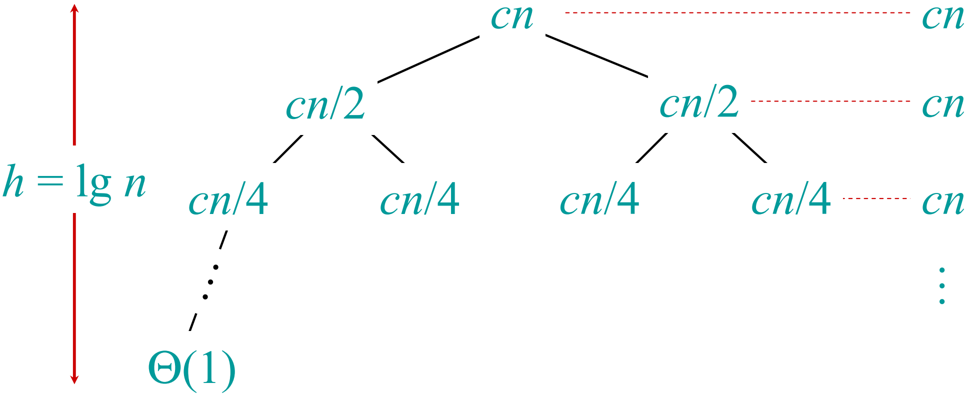

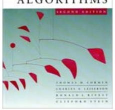

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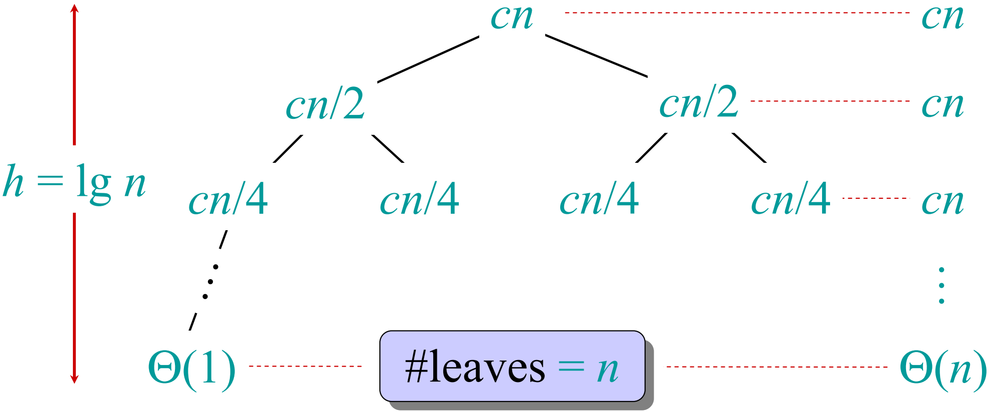

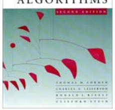

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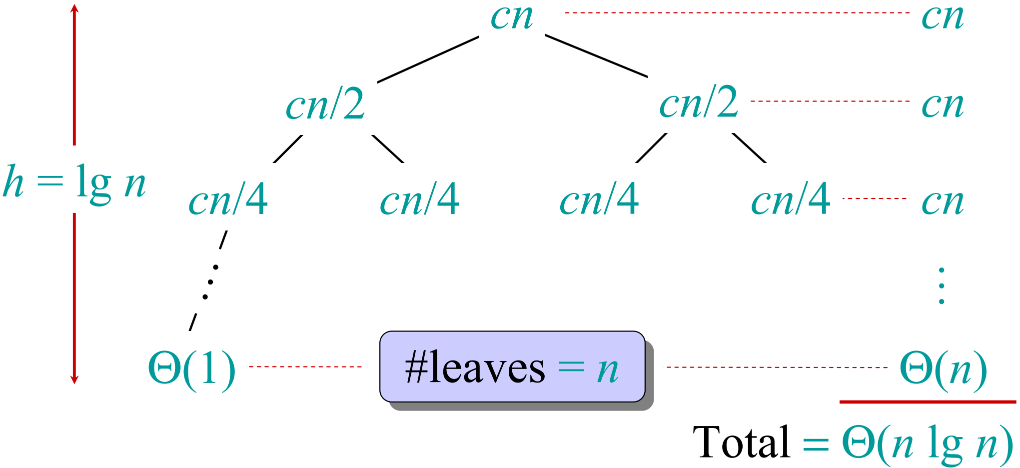

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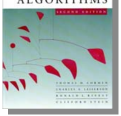
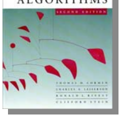


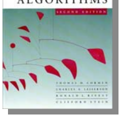
# Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.
- The recursion tree method is good for generating guesses for the substitution method.



# Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



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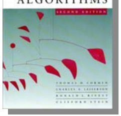

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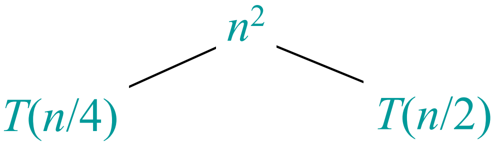

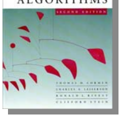

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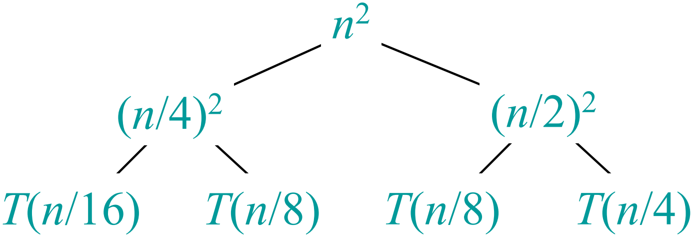

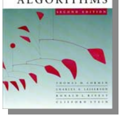

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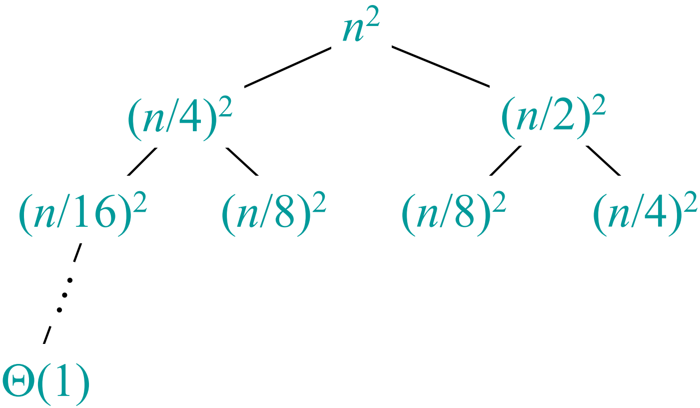

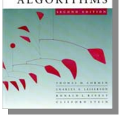

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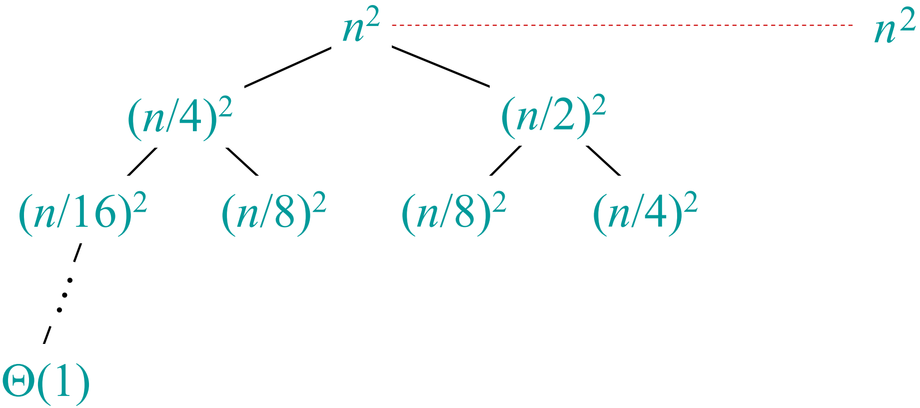

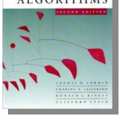

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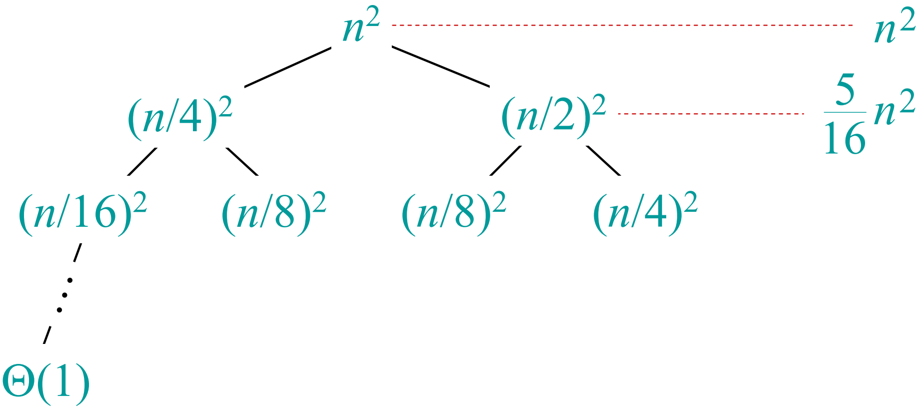
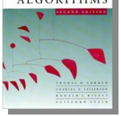
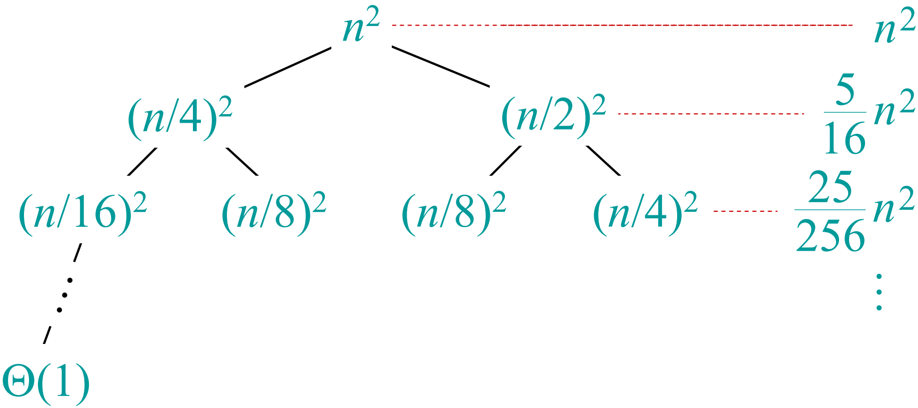
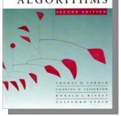




# Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:





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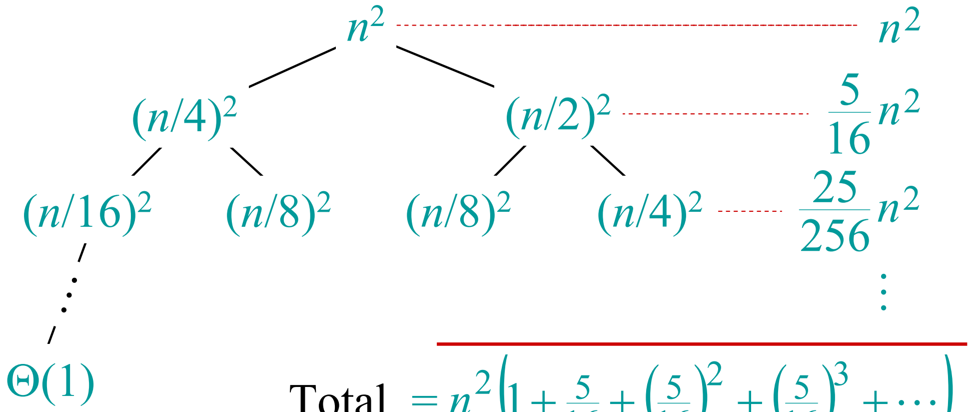
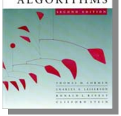



Total $= n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$

$= \Theta(n^2)$ *geometric series*



# Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

Return to last slide viewed.

# extra example

$$T(n) = 2T(3n/8) + n^2$$

- board

# Contents

Divide and Conquer
- Intro
- Merge Sort
- Recursion Tree for recurrences
- Integer Multiplication

# Integer Multiplication

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Could we do better if we used results from subinstances?

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# Integer Multiplication

A Divide and Conquer approach can be considered to be a very large scale version of multiplication, only using base $2^{\lfloor \frac{n}{2} \rfloor}$ instead of a constant base.

For simplicity we assume that $n$ is a power of 2, so $\frac{n}{2}$ will always be integer.

So we split the bitstring for each of $x$ and $y$ in half, generating $x_L$, $x_R$, $y_L$, $y_R$ such that

$$x = 2^{\frac{n}{2}} \cdot x_L + x_R$$
$$y = 2^{\frac{n}{2}} \cdot y_L + y_R\,.$$

Using $x_L, x_R, y_L, y_R$ we can now express our multiplication of the $n$-bit integers as *four* multiplications of $\frac{n}{2}$-bit integers:

$$
\begin{aligned}
x \cdot y &= (2^{\frac{n}{2}} \cdot x_L + x_R) \cdot (2^{\frac{n}{2}} \cdot y_L + y_R) \\
&= 2^n \cdot x_L y_L + 2^{\frac{n}{2}} \cdot (x_L y_R + x_R y_L) + x_R y_R
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since the operations to split the numbers and put the multiplication results together all take time linear in the number of bits:

- the numbers are split by bitshifting
- combining the recursion results takes three addition operations and two bitshifts, all linear

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$$= \; 2cn \cdot n - cn \; = \; 2cn^2 - cn \in \Theta(n^2)$$

# Gauss's Method

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$$(x_L + x_R)(y_L + y_R) = x_L y_L + x_L y_R + x_R y_L + y_R x_R$$

we get that

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we get that

$$x_L y_R + x_R y_L \;=\; (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$$

This might be better because we already compute $x_L y_L$ and $x_R y_R$

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- first compute $x_L, x_R, y_L, y_R$ and $x_L + x_R, y_L + y_R$ in linear time

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Except that's not quite right. What we actually have is

$$T(n) = 2T\left(\frac{n}{2}\right) + T(\frac{n}{2} + 1) + O(n)$$

# Solving the recurrence (board)

$$T(n) = 2T\left(\frac{n}{2}\right) + T(\frac{n}{2} + 1) + O(n)$$

Figure 2_2 in DPV

- Does the $+1$ make any difference? Probably not, but how to be sure?