CS3383 Lecture 1.2: The Master Theorem with applications

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Contents

Generic divide and conquer algorithm

function $\textsc{Solve}$(P)
    if $|P|$ is small **then**
        SolveDirectly(P)
    else
        $P_1 \dots P_k = \text{Partition}(P)$
        for $i = 1 \dots k$ **do**
            $S_i = \text{Solve}(P_i)$
        end for
        Combine($S_1 \dots S_k$)
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- How many times do we recurse?
- what fraction of input in each subproblem?
- How much time to combine results?

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- e.g. one call of $\frac{1}{3}$ and one call of $\frac{2}{3}$,
- partition+combine step $\Theta(n \log n)$.

The Master Theorem

If $\exists$ constants $b > 0$, $s > 1$ and $d \geq 0$ such that $T(n) = b \cdot T(\lceil \frac{n}{s} \rceil) + \Theta(n^d)$, then

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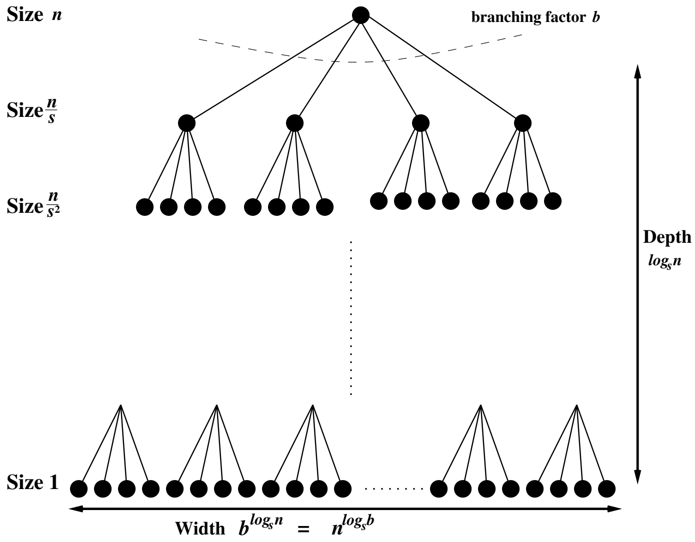
A proof of this follows.

Proof of Master theorem, in pictures



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And so

$$T(n) = \sum_{i=0}^{\log_s n} c \cdot \left(\frac{n}{s^i}\right)^d \cdot b^i$$

Proof of Master theorem, $b = s^d$

$$T(n) = \sum_{i=0}^{\log_s n} c \cdot \left(\frac{n^d}{\left(s^d\right)^i} \right) \cdot b^i \; = \; c \cdot n^d \cdot \left(\sum_{i=0}^{\log_s n} \left(\frac{b}{s^d} \right)^i \right)$$

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If $b = s^d$, then

$$T(n) = c \cdot n^d \cdot \left(\sum_{i=0}^{\log_s n} 1 \right) \;=\; c \cdot n^d \log_s n$$

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$$T(n) = c \cdot n^d \cdot \left(\sum_{i=0}^{\log_s n} 1\right) \;=\; c \cdot n^d \log_s n$$

so $T(n)$ is $\Theta(n^d \log n)$.

Proof of Master Theorem $b \neq s^d$ (1 of 2)

Otherwise ($b \neq s^d$), we have a geometric series,

$$T(n) = c \cdot n^d \cdot \left(\frac{\left(\frac{b}{s^d}\right)^{\log_s n+1} - 1}{\frac{b}{s^d} - 1} \right)$$

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$$T(n) = \frac{s^d}{b - s^d} \cdot c \cdot n^d \cdot \left(\left(\frac{b}{s^d}\right)^{\log_s n+1} - 1 \right)$$

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$$= \frac{s^d}{b - s^d} \cdot c \cdot n^d \cdot \left(\frac{b}{s^d}\right)^{\log_s n+1} - \frac{s^d}{b - s^d} \cdot c \cdot n^d$$

Proof of Master Theorem $b \neq s^d$ (2 of 2)

From rules of powers and logarithms:

$$\left(\frac{b}{s^d}\right)^{\log_s n+1} = \frac{b}{s^d} \cdot \left(\frac{b}{s^d}\right)^{\log_s n} = \frac{b}{s^d} \cdot \frac{b^{\log_s n}}{\left(s^d\right)^{\log_s n}}$$

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Branching versus subproblem size

$$T(n) = \frac{b}{b - s^d} \cdot c \cdot n^{\log_s b} - \frac{s^d}{b - s^d} \cdot c \cdot n^d$$

Now we need to test b versus s^d.

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Sanity check: Merge sort

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Merge Sort

- $T(n) = bT(n/s) + \theta(n^d)$
- b how many recursive calls?
- s what is the the split (denominator of size)
- d degree

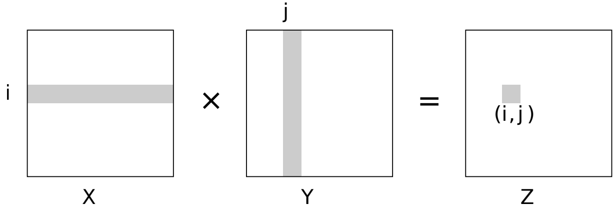
Contents

Matrix Multiplication

The product of two $n \times n$ matrices x and y is a third $n \times n$ matrix $Z = XY$, with

$$Z_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj}$$

where Z_{ij} is the entry in row i and column j of matrix Z.



Calculating Z directly using this formula takes $\Theta(n^3)$ time.

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Eight subinstances $AE, BG, AF, BH, CE, DG, CF, DH$

Matrix Multiplication: Blocks

Recursing 8 times on subinstances of dimension $\frac{n}{2}$, and taking cn^2 time to add the results, gives the time recurrence:

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(this is not technically "cubic algorithm", input size n^2.)

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where

$$\begin{aligned} P_1 &= A(F - H) & P_5 &= (A + D)(E + H) \\ P_2 &= (A + B)H & P_6 &= (B - D)(G + H) \\ P_3 &= (C + D)E & P_7 &= (A - C)(E + F) \\ P_4 &= D(G - E) & & \end{aligned}$$

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Since the input size is $m = n^2$, the algorithm runs in approximately $\Theta(m^{1.404})$ time (versus the $\Theta(m^{1.5})$ of the original).