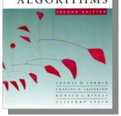
CS3383 Lecture 1.3: Substitution method and randomized d&c

David Bremner

January 19, 2018

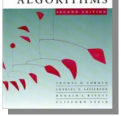
Contents



Substitution method

The most general method:

1. ***Guess*** the form of the solution.
2. ***Verify*** by induction.
3. ***Solve*** for constants.



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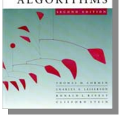
EXAMPLE: $T(n) = 4T(n/2) + n$

- [Assume that $T(1) = \Theta(1)$.]
- Guess $O(n^3)$. (Prove O and Ω separately.)
- Assume that $T(k) \le ck^3$ for $k < n$.
- Prove $T(n) \le cn^3$ by induction.

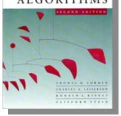


Example of substitution

$$
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^3 + n \\
&= (c/2)n^3 + n \\
&= cn^3 - ((c/2)n^3 - n) \leftarrow \textit{desired} - \textit{residual} \\
&\le cn^3 \leftarrow \textit{desired}
\end{aligned}
$$

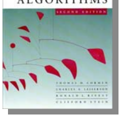
whenever $(c/2)n^3 - n \ge 0$, for example,
if $c \ge 2$ and $n \ge 1$.

residual



Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.



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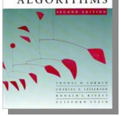
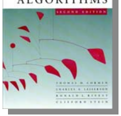
This bound is not tight!



A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.



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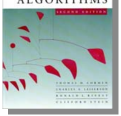
Assume that $T(k) \le ck^2$ for $k < n$:

$$\begin{aligned}
T(n) &= 4T(n/2) + n \\
&\le 4c(n/2)^2 + n \\
&= cn^2 + n \\
&= O(n^2)
\end{aligned}$$



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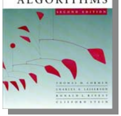
&= cn^2 + n \\
&= \cancel{O(n^2)}
\end{aligned}
$$

Wrong! We must prove the I.H.



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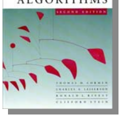
\end{aligned}
$$

Wrong! We must prove the I.H.

$$= cn^2 - (-n) \quad [\text{ desired} - \text{residual }]$$

$$\le cn^2$$ for ***no*** choice of $c > 0$. Lose!

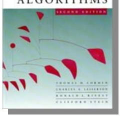


A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1 k^2 - c_2 k$ for $k < n$.



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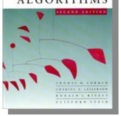
\begin{aligned}
T(n) &= 4T(n/2) + n \\
&= 4(c_1(n/2)^2 - c_2(n/2)) + n \\
&= c_1n^2 - 2c_2n + n \\
&= c_1n^2 - c_2n - (c_2n - n) \\
&\le c_1n^2 - c_2n \quad \text{if } c_2 \ge 1.
\end{aligned}
$$



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&= c_1 n^2 - c_2 n - (c_2 n - n) \\
&\le c_1 n^2 - c_2 n \ \text{ if } c_2 \ge 1.
\end{aligned}$$

Pick c_1 big enough to handle the initial conditions.

More examples of solving recurrences by induction (board)

- $T(n) = T(n-1) + c^n$
- $T(n) = T(n/5) + T(3n/4) + \Theta(n)$

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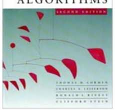
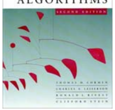


Quicksort

- Proposed by C.A.R. Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).



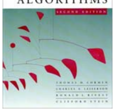
Divide and conquer

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*



Partitioning subroutine

PARTITION(A, p, q) $\triangleright$ $A[p\,.\,.\,q]$

    $x \leftarrow A[p]$ $\triangleright$ pivot $= A[p]$

    $i \leftarrow p$

    for $j \leftarrow p + 1$ **to** q

        do if $A[j] \le x$

            then $i \leftarrow i + 1$

                exchange $A[i] \leftrightarrow A[j]$

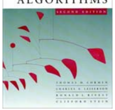
    exchange $A[p] \leftrightarrow A[i]$

    return i

Running time $= O(n)$ for n elements.

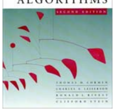
Invariant:

x	$\le x$	$\ge x$	?
p	i	j	q



Example of partitioning

6	10	13	5	8	3	2	11
i	j						



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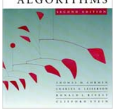

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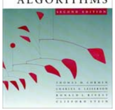
i *j*



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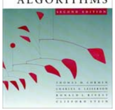

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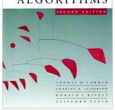

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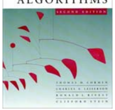
i $\longrightarrow j$



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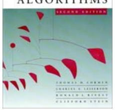

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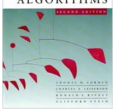
		i				j	



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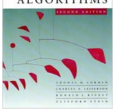
6	5	3	2	8	13	10	11

$\longrightarrow i$ $\qquad\qquad j$



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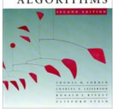
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11

i j



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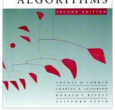

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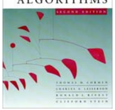
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
2	5	3	6	8	13	10	11
			i				



Pseudocode for quicksort

QUICKSORT(A, p, r)

  if $p < r$

    then $q \leftarrow$ PARTITION(A, p, r)

      QUICKSORT$(A, p, q-1)$

      QUICKSORT$(A, q+1, r)$

Initial call: QUICKSORT$(A, 1, n)$

Analysis of quicksort

- Quicksort is $\Theta(n^2)$ in the worst case. What kind of input is bad?
- Quicksort is supposed to be fast ”in practice”.
- We can choose a better pivot in $O(n)$ time, but we'll see it's a bit complicated.
- What if we choose a random element as pivot?

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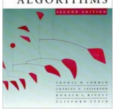
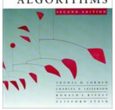


Randomized quicksort

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.



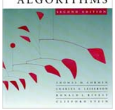
Randomized quicksort analysis

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if } \textsc{Partition} \text{ generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

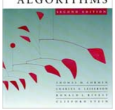
$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.



Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

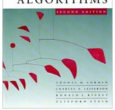
$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right)$$



Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

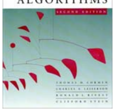


Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

Linearity of expectation.



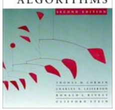
Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)]$$

Independence of X_k from other random choices.



Calculating expectation

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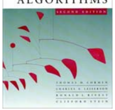
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1}\Theta(n)
\end{aligned}
$$

Linearity of expectation; $E[X_k] = 1/n$.



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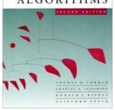
&= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1}\Theta(n) \\
&= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)
\end{aligned}
$$

Summations have identical terms.



Hairy recurrence

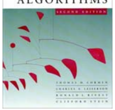
$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

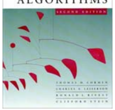
Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).



Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

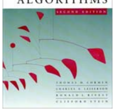


Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.



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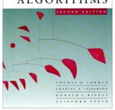
$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired*** – ***residual***.

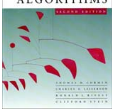


Substitution method

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n) \\
&= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n) \\
&= an\lg n - \left(\frac{an}{4} - \Theta(n)\right) \\
&\le an\lg n,
\end{aligned}
$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.

Contents



Order statistics

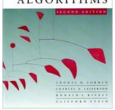
Select the ith smallest of n elements (the element with ***rank*** i).

- $i = 1$: ***minimum***;
- $i = n$: ***maximum***;
- $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: ***median***.

Naive algorithm: Sort and index ith element.

Worst-case running time $= \Theta(n \lg n) + \Theta(1)$
$= \Theta(n \lg n)$,

using merge sort or heapsort (*not* quicksort).



Randomized divide-and-conquer algorithm

RAND-SELECT(A, p, q, i) $\triangleright$ ith smallest of $A[p\,..\,q]$

  if $p = q$ **then return** $A[p]$

  $r \leftarrow$ RAND-PARTITION(A, p, q)

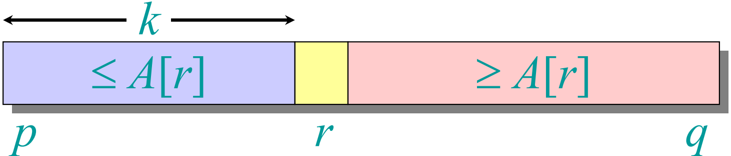
  $k \leftarrow r - p + 1$ $\triangleright$ $k = \mathrm{rank}(A[r])$

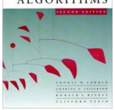
  if $i = k$ **then return** $A[r]$

  if $i < k$

    then return RAND-SELECT$(A, p, r - 1, i)$

    else return RAND-SELECT$(A, r + 1, q, i - k)$





Example

Select the $i = 7$th smallest:

6	10	13	5	8	3	2	11

$i = 7$

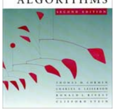
pivot

Partition:

2	5	3	6	8	13	10	11

$k = 4$

Select the $7 - 4 = 3$rd smallest recursively.



Intuition for analysis

(All our analyses today assume that all elements are distinct.)

Lucky:

$$T(n) = T(9n/10) + \Theta(n) \qquad n^{\log_{10/9} 1} = n^0 = 1$$

$$= \Theta(n) \qquad \text{CASE 3}$$

Unlucky:

$$T(n) = T(n-1) + \Theta(n) \qquad \text{arithmetic series}$$

$$= \Theta(n^2)$$

Worse than sorting!

Analysis of randomized median finding (board)

```
Select2(A, p, q, i)
    n <- q - p + 1
    do {
        r <- RandPartition(A,p,q)
        k <- r - p + 1
        if i = k then return A[r]
    } while ((k < n/4) or (k > 3n/4));
    if i < k
        then return Select2 (A, p, r - 1, i )
        else return Select2 (A, r + 1, q, i - k )
```

- Call a pivot r good if $\lfloor n/4 \rfloor$ elements are on either side.
- Odds are 50/50.

Contents

Deterministically choosing a good pivot.

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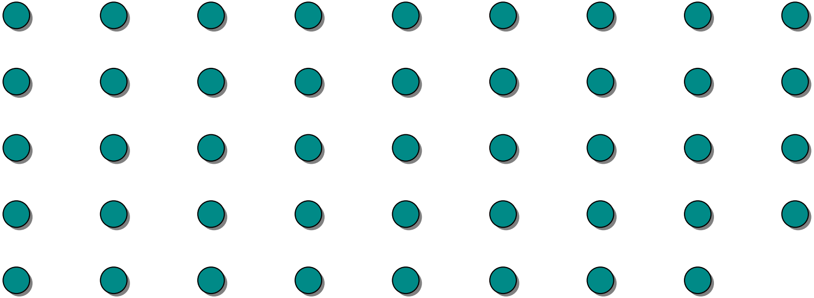
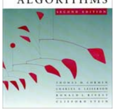
Deterministically choosing a good pivot.

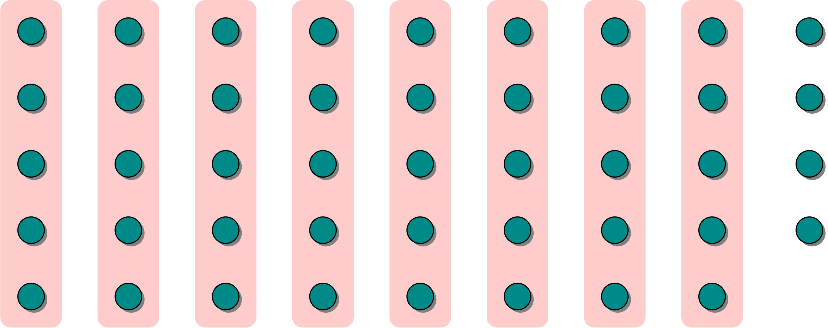
- it turns out we can achieve asymptotically the same worst case time as expected (although worse practically)
- The deterministic algorithm is also more complicated; this is typically one of the main attractions of randomized algorithms, simplicity
- The main idea of this algorithm is taking the median of medians

Choosing the pivot

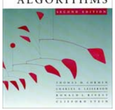




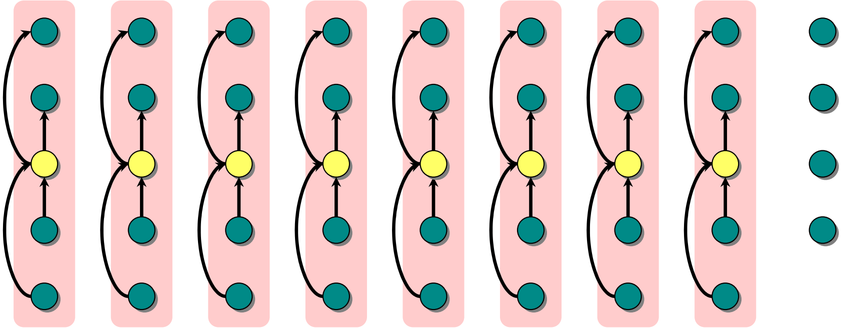
Choosing the pivot



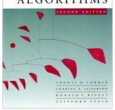
1. Divide the n elements into groups of 5.



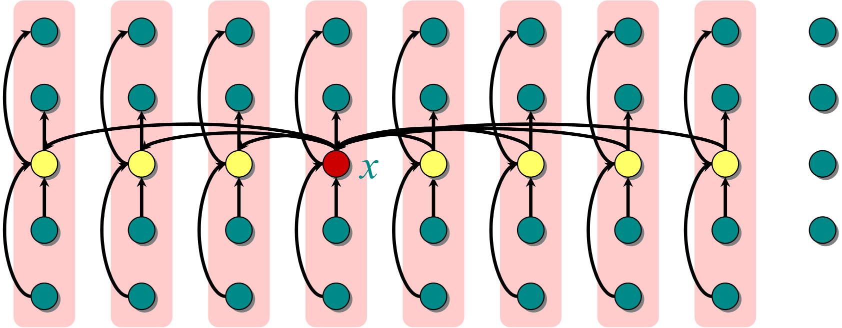
Choosing the pivot



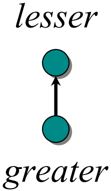
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

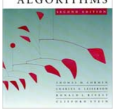


Choosing the pivot

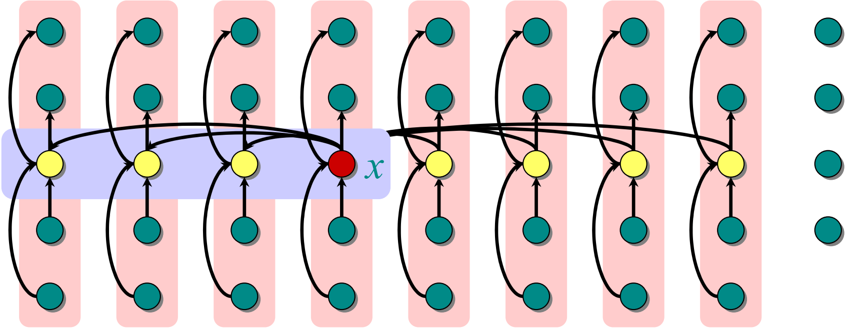


1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

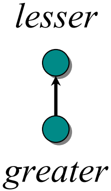


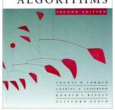


Analysis

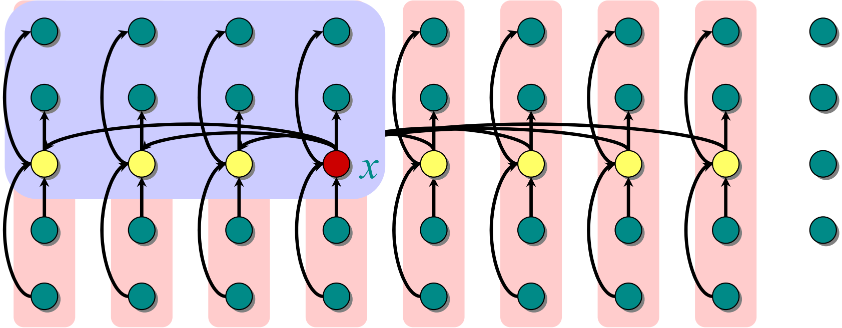


At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.



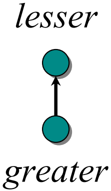


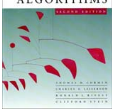
Analysis (Assume all elements are distinct.)



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3 \lfloor n/10 \rfloor$ elements are $\leq x$.





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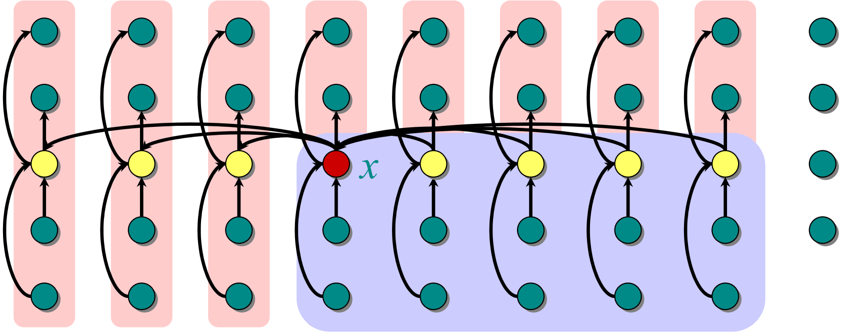

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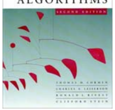
- Therefore, at least $3 \lfloor n/10 \rfloor$ elements are $\le x$.
- Similarly, at least $3 \lfloor n/10 \rfloor$ elements are $\ge x$.

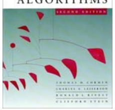
lesser

greater



Minor simplification

- For $n \geq 50$, we have $3\lfloor n/10 \rfloor \geq n/4$.
- Therefore, for $n \geq 50$ the recursive call to SELECT in Step 4 is executed recursively on $\leq 3n/4$ elements.
- Thus, the recurrence for running time can assume that Step 4 takes time $T(3n/4)$ in the worst case.
- For $n < 50$, we know that the worst-case time is $T(n) = \Theta(1)$.



Developing the recurrence

$T(n)$	SELECT(i, n)
$\Theta(n)$	1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
$T(n/5)$	2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
$\Theta(n)$	3. Partition around the pivot x. Let $k = \text{rank}(x)$.
$T(3n/4)$	4. **if** $i = k$ **then return** x **elseif** $i < k$ **then** recursively SELECT the ith smallest element in the lower part **else** recursively SELECT the $(i–k)$th smallest element in the upper part

Luckily we already solved this recurrence

- $T(n) \leq T(n/5) + T(3n/4) + \Theta(n)$