CS3383 Unit 3.2: Dynamic Programming Examples

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Outline

- Dynamic Programming
  - Longest increasing subsequence
  - Edit Distance

Contents

Longest increasing subsequence problem

Input Integers $a_1, a_2 \ldots a_n$

Output

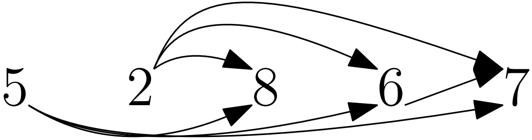
$$a_{i_1}, a_{i_2}, \ldots a_{i_k}$$

Such that

$$i_1 < i_2 \cdots < i_k$$

and

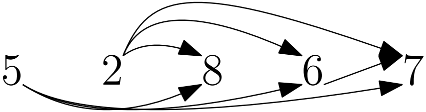
$$a_{i_1} < a_{i_2} < \cdots < a_{i_k}$$



- $(a_i, a_j) \in E$ if $i < j$ and $a_i < a_j$.
- DPV 6.2, JE 5.2

Defining subproblems

- Define $F(i)$ as the length of longest sequence starting at position i



$$F(i) = 1 + \max\{F(j) \mid (i,j) \in E\}$$

- Topological sort is trivial

- We could solve this reasonably fast e.g. by memoization.

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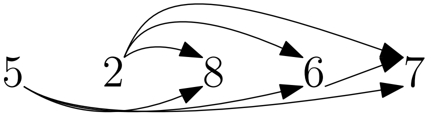

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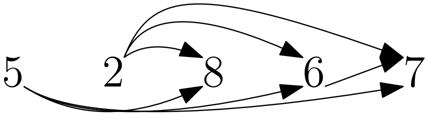
Defining subproblems

- Define $F(i)$ as the length of longest sequence starting at position i
- We could do n longest path in DAG queries.
- Thinking recursively:

$$F(i) = 1 + \max\{F(j) \mid (i, j) \in E\}$$

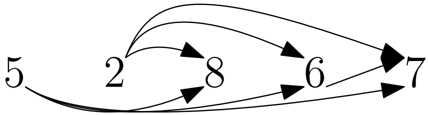
- We could solve this reasonably fast e.g. by memoization.



- Topological sort is trivial

Longest path in DAG, working backwards

- Define $L[i]$ as the longest path *ending* at a_i



```
For i = 1…n:
    L[i] = 1 + max { L(j) | (j,i) in E }
```

- total cost is $O(|E|)$, *after* computing E.

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```
function $\mathrm{LIS}(a_1 \ldots a_n)$
    $\forall i\ L[i] = -\infty$
    for $i \in 1 \ldots n$ do
        for $j \in 1 \ldots i-1$ do
            if $a_j < a_i$ then
                $L[i] \leftarrow \max(L[i], L[j]+1)$
            end if
        end for
    end for
    return $\max(L[1] \ldots L[n])$
end function
```

Contents

Edit (Levenshtein) Distance

- DPV 6.3, JE5.5
- Minimum number of insertions, deletions, substitutions to transform one string into another.

Example: timberlake → fruitcake

- Using mostly insertions and deletions

```
i i i i   d d d d d s
_ _ _ _ T I M B E R L A K E
F R U I T _ _ _ _ _ C A K E
```

Total cost 10.

Edit (Levenshtein) Distance

- DPV 6.3, JE5.5
- Minimum number of insertions, deletions, substitutions to transform one string into another.

Example: timberlake → fruitcake

- Using more substitutions

```
s s s s s d s
T I M B E R L A K E
F R U I T _ C A K E
```

Total cost 7.

Alignments (gap representation)

```
1 1 1 1 0 1 1 1 1 1 1 0 0 0
_ _ _ _ T I M B E R L A K E
F R U I T _ _ _ _ _ C A K E
```

- top line has letters from A, in order, or `_`
- bottom line has has letters from B or `_`
- cost per column is 0 or 1.

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Theorem (Optimal substructure)

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proof.

By contradiction □

Subproblems (prefixes)

- Define $E[i, j]$ as the minimum edit cost for $A[1 \dots i]$ and $B[1 \dots j]$

$$E[i,j] = \begin{cases} E[i, j-1] + 1 & \text{insertion} \\ E[i-1, j] + 1 & \text{deletion} \\ E[i-1, j-1] + 1 & \text{substition} \\ E[i-1, j-1] & \text{equality} \end{cases}$$

justification.

We know deleting a column removes an element from one or both strings; all edit operations cost 1. ☐

order of subproblems

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- dependency of subproblems is *exactly* the same as LCS, so essentially the same DP algorithm works.
- or just memoize the recursion
- what are the base cases?