CS3383 Unit 5: Backtracking and SAT

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Outline

- Combinatorial Search
  - Backtracking
  - SAT
  - Tractable kinds of SAT

Contents

N-queens

Problem Description

Given an $n \times n$ chess board, can you place n queens so that no two are in the same row, column, or diagonal.

Examples

	Q		
			Q
Q			
		Q	

$\checkmark$

	Q		
			Q
	Q		
		Q	

no

- We can eliminate the problem case on the right immediately

Representing Chessboards

- We only care about cases where there is 1 queen per column
- Represent a $n \times n$ board as an array of n integers, meaning which row.
- 0 for not chosen yet.

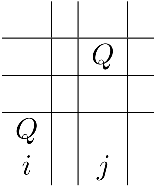
	Q		
			Q
Q			
		Q	

$(2, 4, 1, 3)$

		Q	
Q			

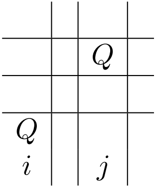
$(4, 0, 2, 0)$

Detecting collisions



$$Q[j] - Q[i] = j - i$$

Detecting collisions



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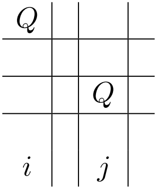

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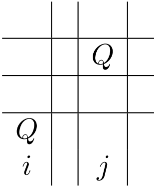

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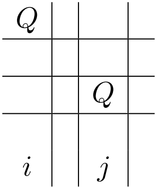

$$Q[j] - Q[i] = i - j$$

- And one more (easy) case

Backtracking Requirements

1. A representation for partial solutions

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1. A representation for partial solutions
2. A procedure to expand a problem into smaller subproblems
3. A test for partial solutions that returns

SUCCESS	if the solution is complete
FAILURE	if there is no way to complete
UNKNOWN	if neither of the above can be quickly determined.

Generic Backtracking

```
function BACKTRACK(P_0)
    S ← { P_0 }
    while ! empty(S) do
        P ← S.dequeue()
        for R ∈ expand(P) do
            switch test(R) do
                case SUCCESS
                    return SUCCESS
                case UNKNOWN
                    S.enqueue(R)
        end for
    end while
end function
```

Backtracking for N-Queens

representation $Q[1...n]$ where $Q[i]$ is row chosen, or 0 for none.

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```
function TEST(Q)
    default ← SUCCESS
    for i ∈ 1 ... n − 1 do
        if Q[i]=0 then
            default ← UNKNOWN
        else
            for j ∈ 1 ... i − 1 do
                if Q[i] − Q[j] ∈ { 0, i − j, j − i } then
                    return FAIL
                end if
            end for
        end if
    end for
    return default
end function
```

Backtracking for subset sum

Subset Sum

Given $X \subset \mathbb{R}+$, T

Decide Is there a subset of X that sums to T

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Backtracking for subset sum

Subset Sum

Given $X \subset \mathbb{R}+$, T

Decide Is there a subset of X that sums to T

Branching

- If (X, T) is feasible for some Z, for all $y \in X$, either the solution includes y or not.

Backtracking for SubsetSum

```
function SubsetSum(X,T)
    if T = 0 then
        return true
    elseif T < 0 or X = ∅
        return false
    end
    (y, X′) ← pop(X)
    return SubsetSum(X′, T − y)
        or SubsetSum(X′, T)
end
```

Contents

The SAT Problem

Conjunctive Normal Form (CNF)

Variables $\{ x_1 ... x_n \}$

Literals $L = \{ x_i, \bar{x}_i \mid \text{variable } x_i \}$

Clauses $\{ z_1, ..., z_k \} \subset L$

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Propositional Satisfiability (SAT)

Instance Set of clauses $\mathcal{C}$

Question Is there an assignment of 0, 1 to every variable such that each clause has at least one true literal?

SAT Example

$$\{\{1,2,3\},\{-1,-2,-3\}\} = \{\{x_1,x_2,x_3\},\{\bar{x}_1,\bar{x}_2,\bar{x}_3\}\}$$
$$=$$

(A)

Truth Table

x_1	x_2	x_3	A
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

SAT Example

$$\{\,\{\,1,2,3\,\},\{\,-1,-2,-3\,\}\,\} = \{\,\{\,x_1,x_2,x_3\,\},\{\,\bar{x}_1,\bar{x}_2,\bar{x}_3\,\}\,\}$$
$$= (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \quad \text{(A)}$$

Truth Table

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Backtracking for SAT

representation (reduced) clauses

test if empty clause, return FAIL. If no clauses, return SUCCESS. Otherwise return UNKNOWN

expand $P_0 = \text{reduce}(P, j, 0)$, $P_1 = \text{reduce}(P, j, 1)$ for some j.

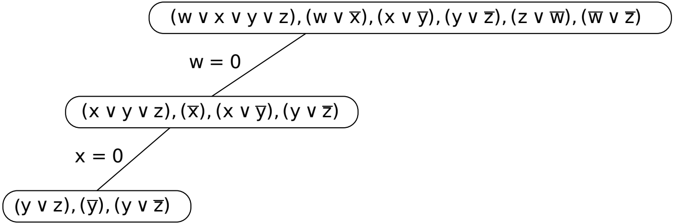
Backtracking SAT Example II

$(w \vee x \vee y \vee z), (w \vee \overline{x}), (x \vee \overline{y}), (y \vee \overline{z}), (z \vee \overline{w}), (\overline{w} \vee \overline{z})$

$w = 0$

$(x \vee y \vee z), (\overline{x}), (x \vee \overline{y}), (y \vee \overline{z})$

Backtracking SAT Example II



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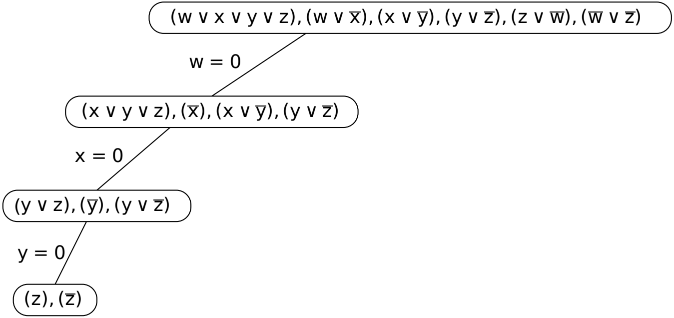

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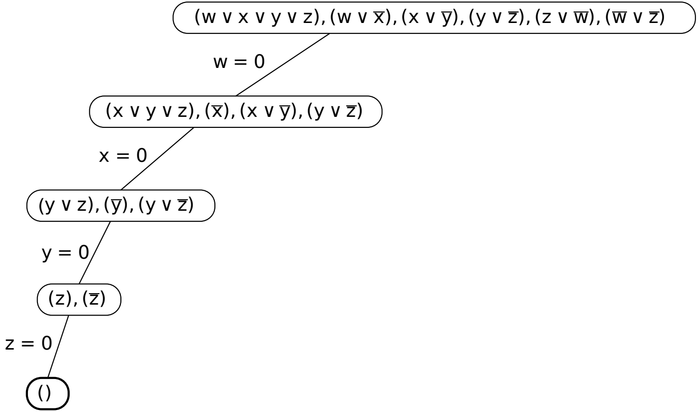

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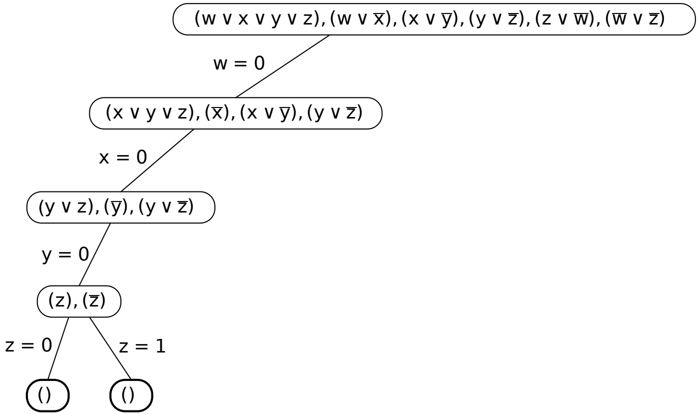

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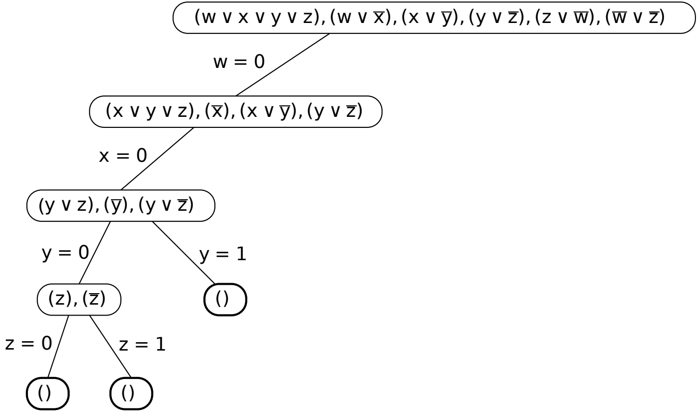

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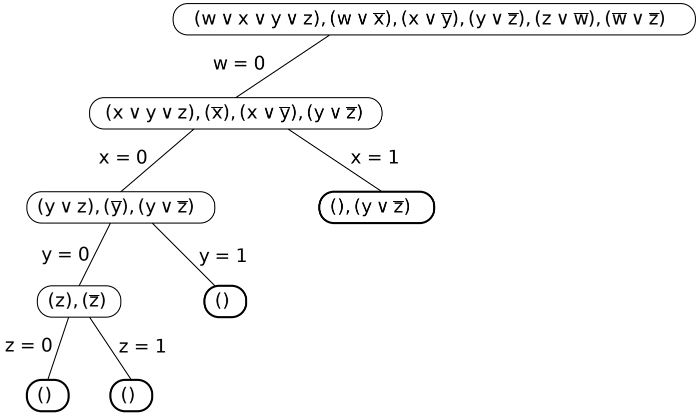

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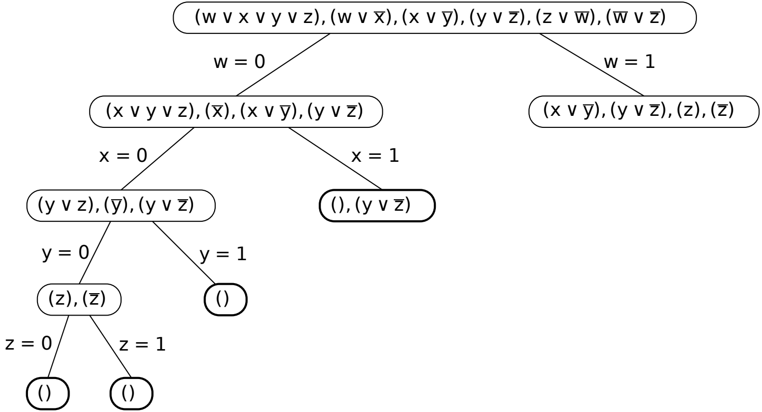

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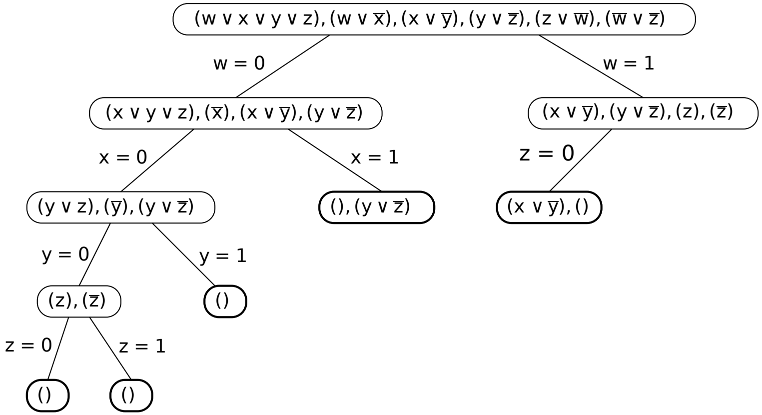

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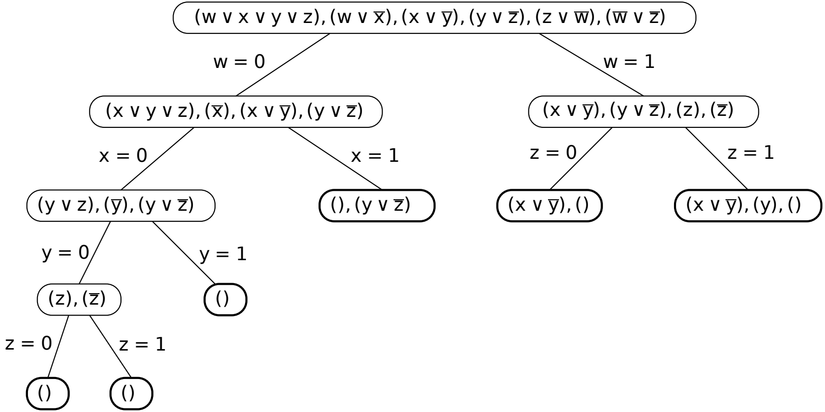

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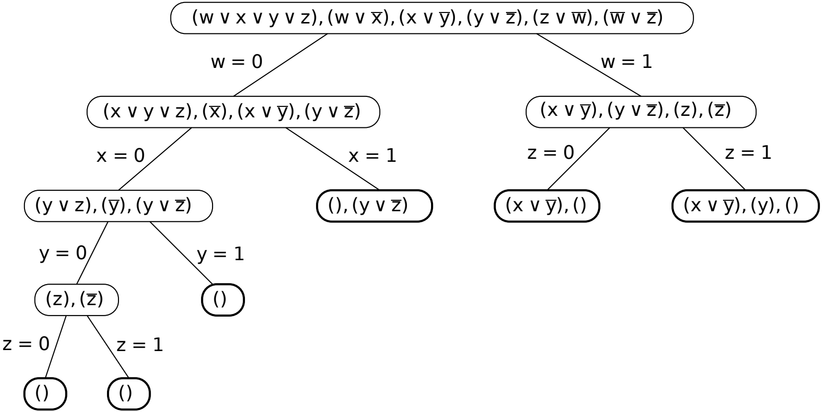

We tried 11 possibilities. What's the maximum possible?

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- to maximize number of clauses satisfied, choose $x_1 \leftarrow 1$, $x_4 \leftarrow 0$
- solvable with *unit propagation*

Horn SAT

Horn formulas

implication $(z \wedge w \wedge q) \Rightarrow u$. LHS is all positive, RHS one positive literal

negative clauses $(\bar{x} \vee \bar{w} \vee \bar{y})$. All literals negated.

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$$(\bigwedge_{i=1}^{k} x_i) \Rightarrow y$$
$$\neg(\bigwedge_{i=1}^{k} x_i) \vee y$$
$$(\bigvee_{i=1}^{k} \bar{x}_i) \vee y$$

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$$\neg(\bigwedge_{i=1}^{k} x_i) \vee y$$

$$(\bigvee_{i=1}^{k} \bar{x}_i) \vee y$$

- So we can think about special CNF with at most one positive literal.

Unit propagation

function UNITPROP(S: Set of clauses)
    while S has a unit clause $C = \{ z \}$ **do**
        if $z = \bar{x}_i$ **then**
            $x_i \leftarrow 0$
        else
            $x_i \leftarrow 1$
        end if
        $S \leftarrow \{ C \mid C \in S, z \notin C \}$
        If $S = \emptyset$, return SATISFIABLE
        $S \leftarrow \{ C \setminus \{ \neg z \} \mid C \in S \}$
        If $\emptyset \in S$, return UNSATISFIABLE
    end while
    return S
end function

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  end while
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end function

- Example: $\{ x_1 \}, \{ \bar{x}_1, x_2 \}, \{ x_2, x_3 \}$

Solving Horn SAT with Unit Propagation

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1. Apply unit propagation
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Proof

any remaining clause has at least one negative literal