

CS3383 Unit 5.1: Backtracking and SAT

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Outline

Combinatorial Search

Backtracking

SAT

Tractable kinds of SAT

Contents

Combinatorial Search

Backtracking

SAT

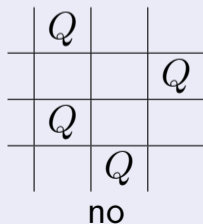
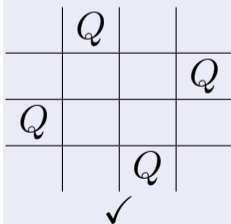
Tractable kinds of SAT

N-queens

Problem Description

Given an $n \times n$ chess board, can you place n queens so that no two are in the same row, column, or diagonal.

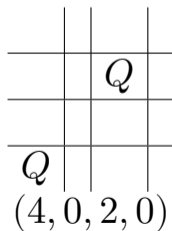
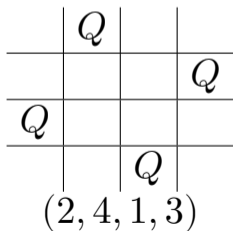
Examples



► We can eliminate the problem case on the right immediately

Representing Chessboards

- ▶ We only care about cases where there is 1 queen per column
- ▶ Represent a $n \times n$ board as an array of n integers, meaning which row.
- ▶ 0 for not chosen yet.



Detecting collisions

		Q	
Q			
i		j	

$$Q[j] - Q[i] = j - i$$

Detecting collisions

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Detecting collisions

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Q		
i		j
Q		
		Q
i		j

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$$Q[j] - Q[i] = i - j$$

► And one more (easy) case

Backtracking Requirements

1. A representation for partial solutions

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1. A representation for partial solutions
2. A procedure to **expand** a problem into smaller subproblems
3. A test for partial solutions that returns
SUCCESS if the solution is complete
FAILURE if there is no way to complete
UNKNOWN if neither of the above can be quickly determined.

Generic Backtracking

```
function BACKTRACK( $P_0$ )  
   $S \leftarrow \{ P_0 \}$   
  while ! empty( $S$ ) do  
     $P \leftarrow S.dequeue()$   
    for  $R \in \text{expand}(P)$  do  
      switch test( $R$ ) do  
        case SUCCESS  
          return SUCCESS  
        case UNKNOWN  
           $S.enqueue(R)$   
      end for  
  end while  
  return FAIL
```

Backtracking for N-Queens: Framework

representation $Q[1\dots n]$ where $Q[i]$ is row chosen, or 0 for none.

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expand For some $Q[i] = 0$, try $Q[i] = 1\dots n$

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```
def test(Q):  
    default  $\leftarrow$  SUCCESS  
    for  $i \in 1\dots n - 1$ :  
        if  $Q[i] = 0$ :  
            default  $\leftarrow$  UNKNOWN  
        else:  
            for  $j \in 1\dots i - 1$ :  
                if  $Q[i] - Q[j] \in \{0, i - j, j - i\}$ :  
                    return FAIL  
    return default
```

Backtracking for subset sum

Subset Sum

Given $X \subset \mathbb{R}_+, T$

Decide Is there a subset of X that sums to T

Backtracking for subset sum

Subset Sum

Given $X \subset \mathbb{R}_+$, T

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Branching

Backtracking for subset sum

Subset Sum

Given $X \subset \mathbb{R}_+, T$

Decide Is there a subset of X that sums to T

Branching

- ▶ If (X, T) is feasible for some Z , for all $y \in X$, either the solution includes y or not.

Backtracking for SubsetSum

```
function SubsetSum( $X, T$ )  
  if  $T = 0$  then  
    return true  
  elseif  $T < 0$  or  $X = \emptyset$   
    return false  
  end  
   $(y, X') \leftarrow \text{pop}(X)$   
  return SubsetSum( $X', T - y$ )  
    or SubsetSum( $X', T$ )  
end
```

Contents

Combinatorial Search

Backtracking

SAT

Tractable kinds of SAT

The SAT Problem

Conjunctive Normal Form (CNF)

Variables $\{x_1 \dots x_n\}$

Literals $L = \{x_i, \bar{x}_i \mid \text{variable } x_i\}$

Clauses $\{z_1, \dots, z_k\} \subset L$

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Propositional Satisfiability (SAT)

Instance Set of clauses \mathcal{C}

Question Is there an assignment of 0, 1 to every variable such that each clause has at least one true literal?

SAT Example

$$\{ \{ 1, 2, 3 \}, \{ -1, -2, -3 \} \} = \{ \{ x_1, x_2, x_3 \}, \{ \bar{x}_1, \bar{x}_2, \bar{x}_3 \} \}$$
$$=$$

(A)

Truth Table

x_1	x_2	x_3	A
0	0	0	
0	0	1	
0	1	0	
	\vdots		

SAT Example

$$\begin{aligned}\{ \{ 1, 2, 3 \}, \{ -1, -2, -3 \} \} &= \{ \{ x_1, x_2, x_3 \}, \{ \bar{x}_1, \bar{x}_2, \bar{x}_3 \} \} \\ &= (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \\ &\qquad\qquad\qquad (A)\end{aligned}$$

Truth Table

x_1	x_2	x_3	A
0	0	0	
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0	1	0	
	\vdots		

Backtracking for SAT

representation (reduced) clauses

test if empty clause, return FAIL. If no clauses, return SUCCESS. Otherwise return UNKNOWN

expand $P_0 = \text{reduce}(P, j, 0)$, $P_1 = \text{reduce}(P, j, 1)$ for some j .

Backtracking SAT Example II

$(w \vee x \vee y \vee z), (w \vee \bar{x}), (x \vee \bar{y}), (y \vee z), (z \vee \bar{w}), (\bar{w} \vee z)$

$w = 0$

$(x \vee y \vee z), (\bar{x}), (x \vee \bar{y}), (y \vee z)$

Backtracking SAT Example II

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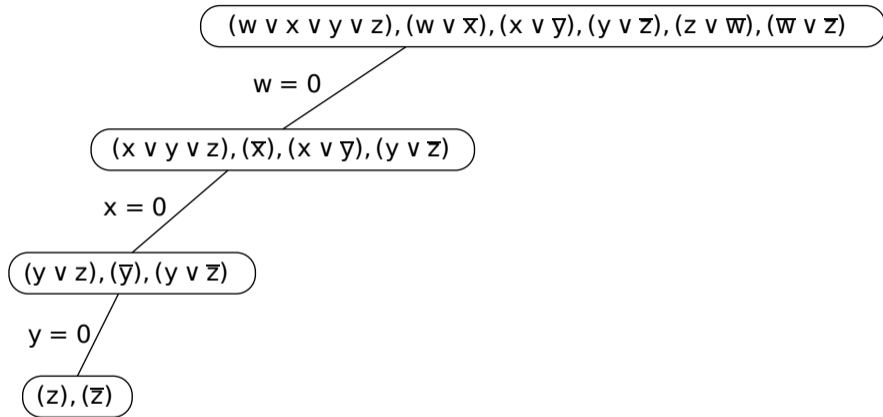
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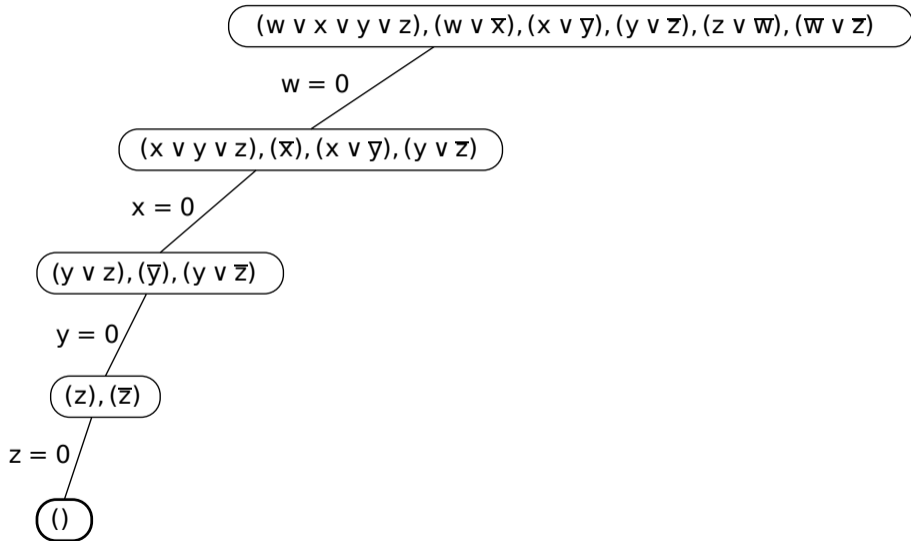
$x = 0$

$(y \vee z), (\bar{y}), (y \vee \bar{z})$

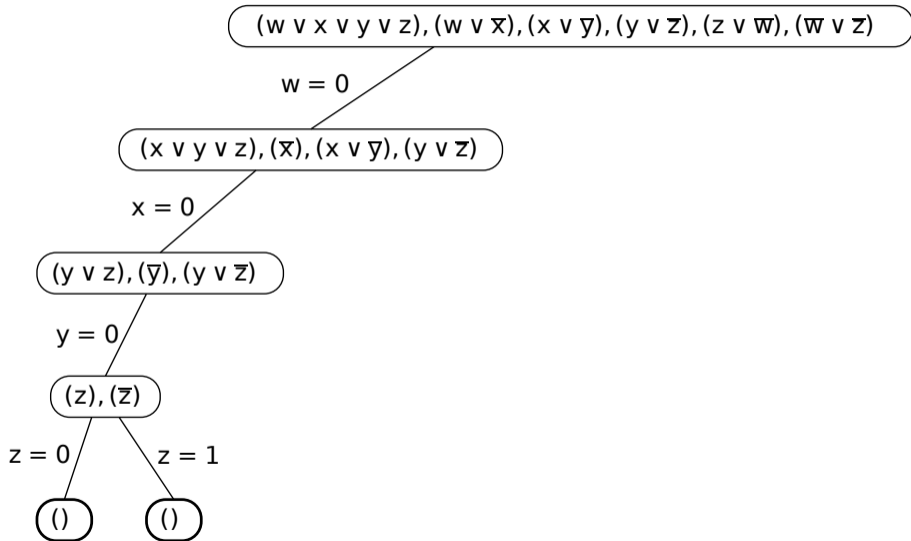
Backtracking SAT Example II



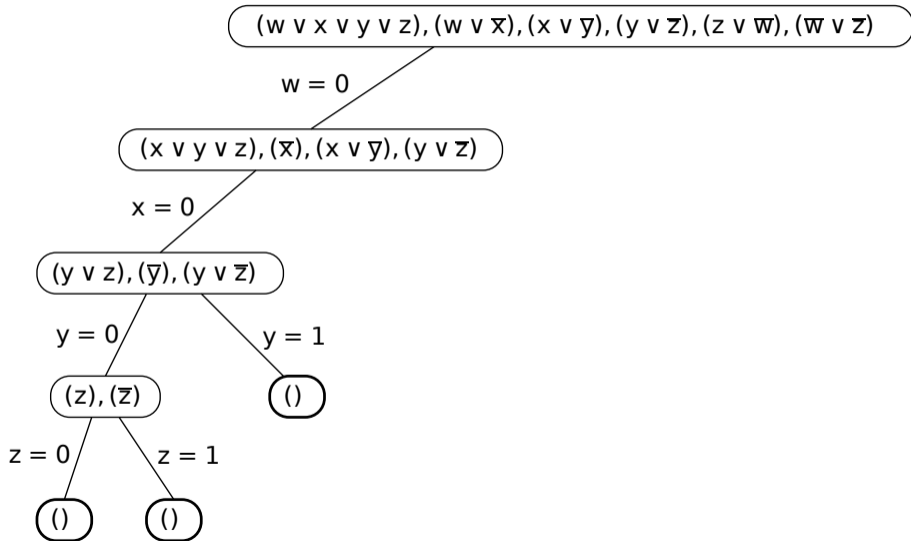
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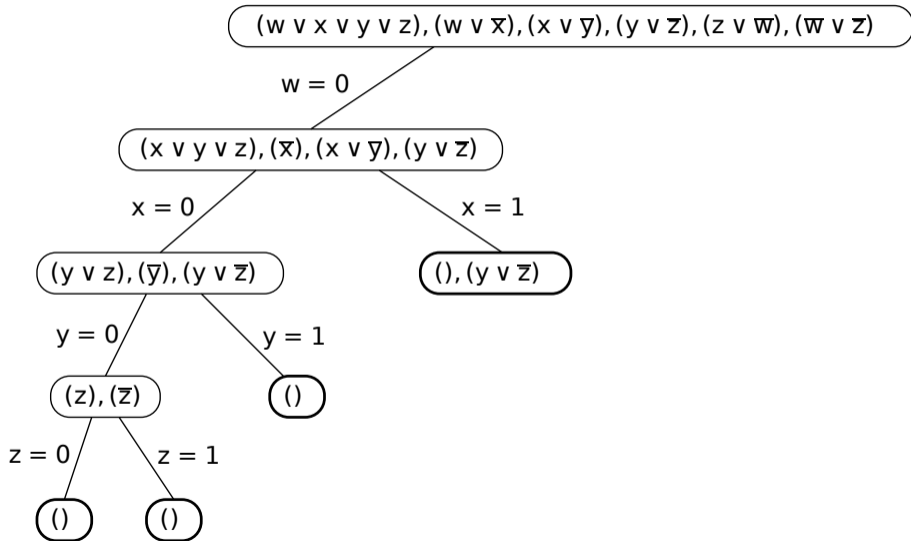
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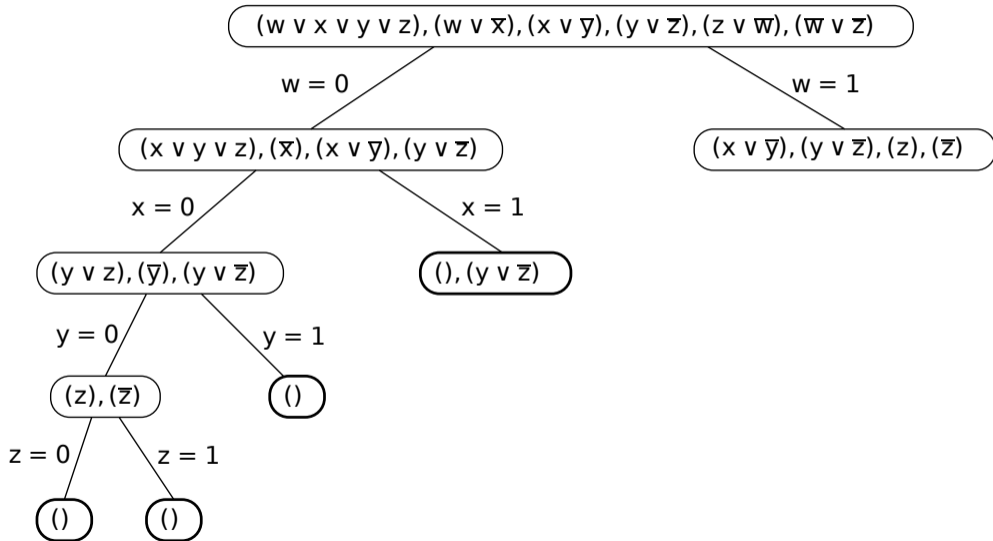
Backtracking SAT Example II



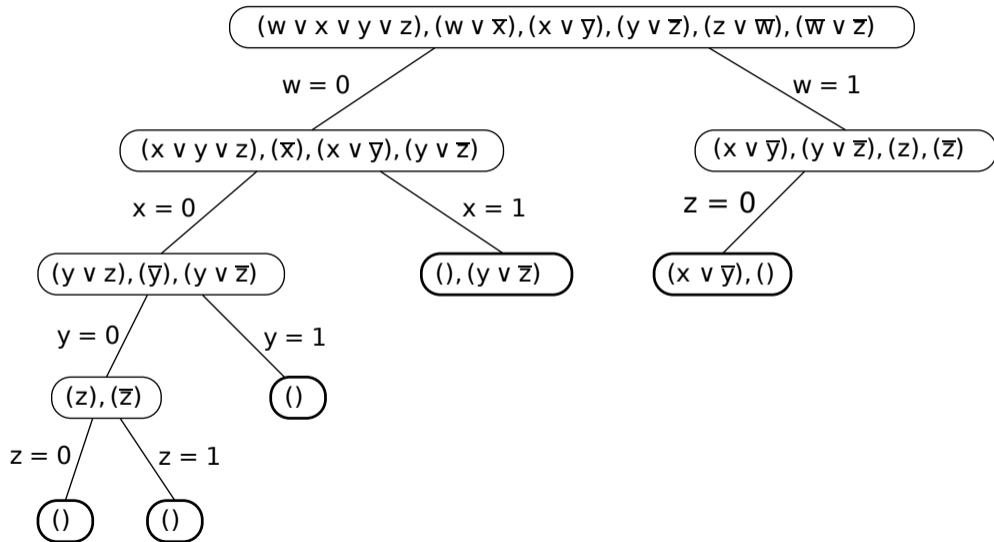
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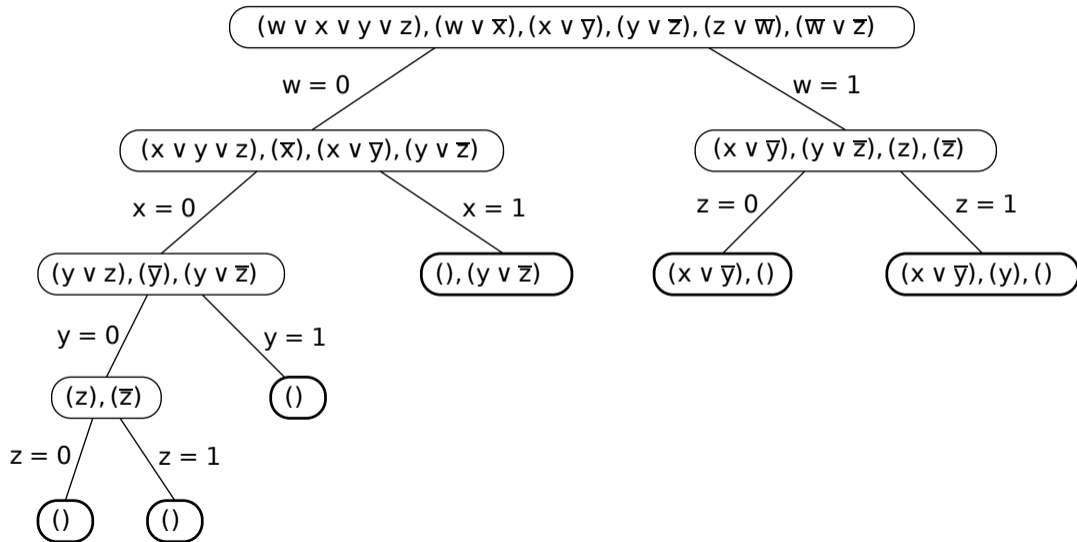
Backtracking SAT Example II



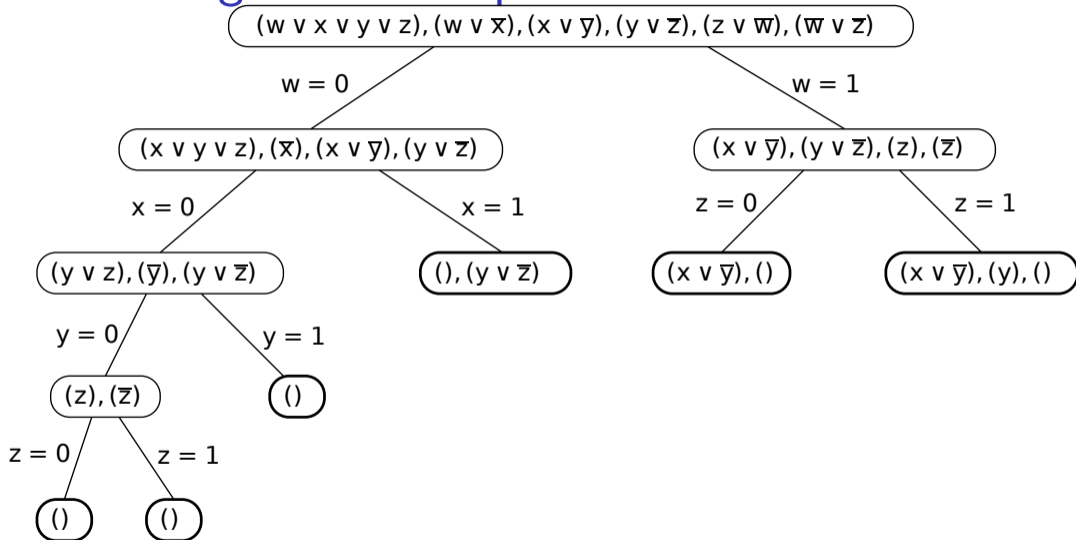
Backtracking SAT Example II



Backtracking SAT Example II



Backtracking SAT Example II



We tried 11 possibilities. What's the maximum possible?

Contents

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SAT

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- ▶ to maximize number of clauses satisfied, choose $x_1 \leftarrow 1$,
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- ▶ solvable with *unit propagation*

Horn SAT

Horn formulas

implication $(z \wedge w \wedge q) \Rightarrow u$. LHS is all positive, RHS one positive literal

negative clauses $(\bar{x} \vee \bar{w} \vee \bar{y})$. All literals negated.

Horn formulas as CNF

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- ▶ So we can think about special CNF with at most one positive literal.

Unit propagation

```
function UNITPROP( $S$ : Set of clauses)
  while  $S$  has a unit clause  $C = \{z\}$  do
    if  $z = \bar{x}_i$  then
       $x_i \leftarrow 0$ 
    else
       $x_i \leftarrow 1$ 
    end if
     $S \leftarrow \{C \mid C \in S, z \notin C\}$ 
    If  $S = \emptyset$ , return SATISFIABLE
     $S \leftarrow \{C \setminus \{\neg z\} \mid C \in S\}$ 
    If  $\emptyset \in S$ , return UNSATISFIABLE
  end while
return  $S$ 
```

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Solving Horn SAT with Unit Propagation

Procedure HornProp

1. Apply unit propagation
2. If no contradiction is detected, set the remaining variables to false.

Claim 1

If the procedure detects a contradiction, the instance is unsatisfiable

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Proof

any remaining clause has at least one negative literal