CS3383 Unit 2: Greedy

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Outline

- Greedy
  - Properties of (optimal) Huffman trees
  - Huffman algorithm

Lecture background

- CLRS4 §15.3
- `https://jeffe.cs.illinois.edu/teaching/algorithms/book/04-greedy.pdf`
  - Huffman Coding is covered in §4.4
- DPV 5.2

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From Data Structures

- heaps
- priority queues

Prefix codes

Symbol	Freq	Codeword
A	70	0
B	3	001
C	20	01
D	37	11

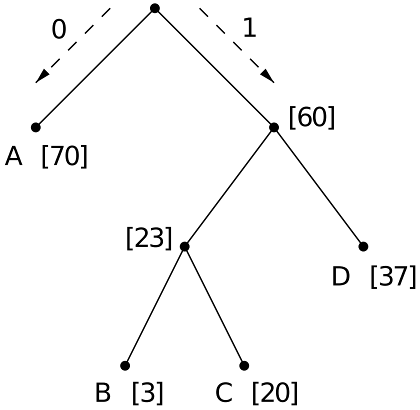
- avoiding ambiguous bitstreams: what is 001?

Prefix codes

Symbol	Freq	Codeword
A	70	0
B	3	100
C	20	101
D	37	11

Huffman coding

Symbol	Codeword
A	0
B	100
C	101
D	11



(Avg cost)

$$\mathrm{cost}(T) = \sum_{i=1}^{n} f_i \mathrm{depth}_i$$

Huffman trees are full

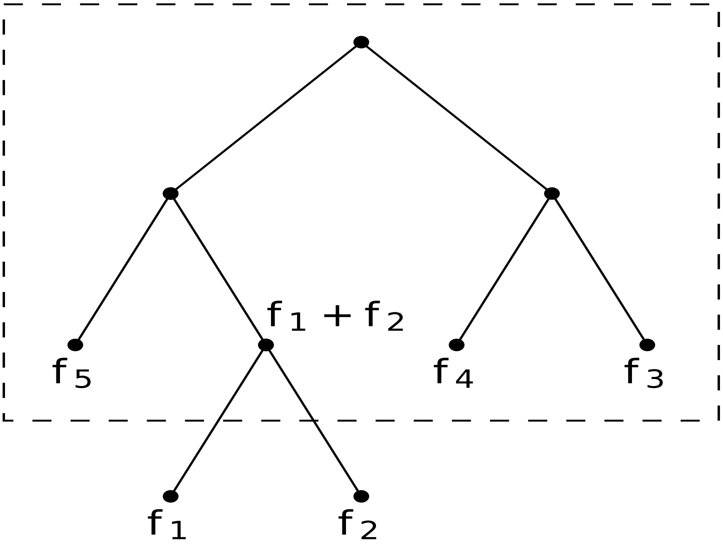
Lemma (Full Trees)

In an optimal Huffman tree every node has zero or two children.

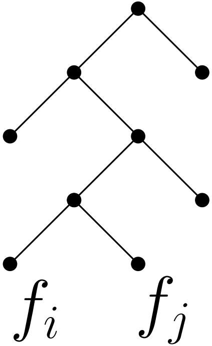
Proof.

If not, consider a node with one child. The corresponding bit in the code can be deleted without changing the property of being a prefix code. $\square$

Lightest leaves are deepest



The two lightest leaves are deepest



Lemma (Lightest siblings)

There exists an optimal tree where the lightest leaves are siblings on the deepest level.

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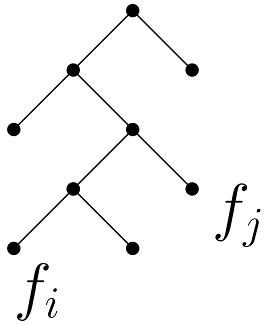


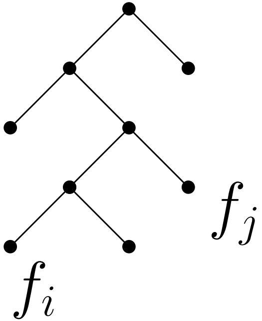
setup for contradiction

Let T be an optimal Huffman tree with $\mathrm{cost}(T) = \sum_k f_k d_k$. Let f_i be a leaf on the deepest level. Suppose some other leaf f_j exists with $f_j < f_i$ and $d_j < d_i$.

The two lightest leaves are deepest

Lemma (Lightest siblings)

There exists an optimal tree where the lightest leaves are siblings on the deepest level.



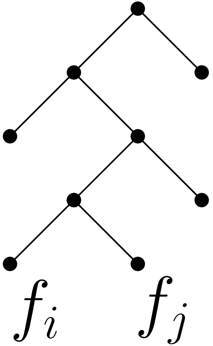
Let T' be the tree with f_i and f_j swapped. The cost of T' is

$$\mathrm{cost}(T') = \left(\sum_k f_k d_k\right) - f_j d_j - f_i d_i + f_j d_i + f_i d_j$$

(residual)

$$= \mathrm{cost}(T) - [(d_j - d_i) \times (f_j - f_i)]$$

The two lightest leaves are deepest



Lemma (Lightest siblings)

There exists an optimal tree where the lightest leaves are siblings on the deepest level.

Once we have all lightest leaves on the bottom level, we can swap them at will without changing the cost of the tree.

Huffman demo

```
def huffman(f):
  tree=[]; H=[]; n = len(f)
  for i in range(0,n):
      heappush(H,(f[i],i))
      tree.append((f[i],None,None))
  for k in range(n,2*n-1):
      (f1,index1) = heappop(H)
      (f2,index2) = heappop(H)
      f3 = f1 + f2
      heappush(H,(f3,k))
      tree.append((f3,index1,index2))
  return tree
```

```
214
   84
      37
      47
         23
            3
           20
         24
   130
      60
      70
```

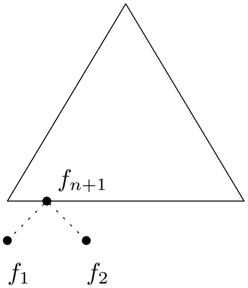
Huffman example

Symbol	Freq	Codeword
A	70	0
B	3	100
C	20	101
D	37	11

- first loop gives us

```
H=[(3,0),(20,1),(37,2),(70,3)]
T=[(3,None,None),(20,None,None),
   (37,None,None),(70,None,None)]
```

Correctness of Huffman algorithm



Theorem (Greedy Huffman Algorithm)

$huffman$ produces an optimal binary prefix code.

Correctness of Huffman algorithm

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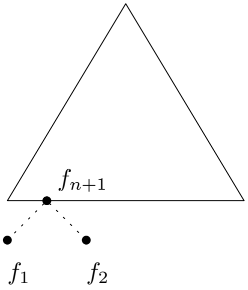
base case

If we have 1 or 2 symbols, any one bit code is optimal.



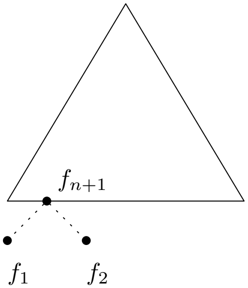
Correctness of Huffman algorithm

Theorem (Greedy Huffman Algorithm)

$huffman$ produces an optimal binary prefix code.

induction hypothesis

For all $k < n$, `huffman(`$[f_1 \ldots f_k]$`)` produces an optimal Huffman tree.



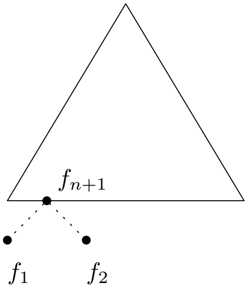
Correctness of Huffman algorithm

Theorem (Greedy Huffman Algorithm)

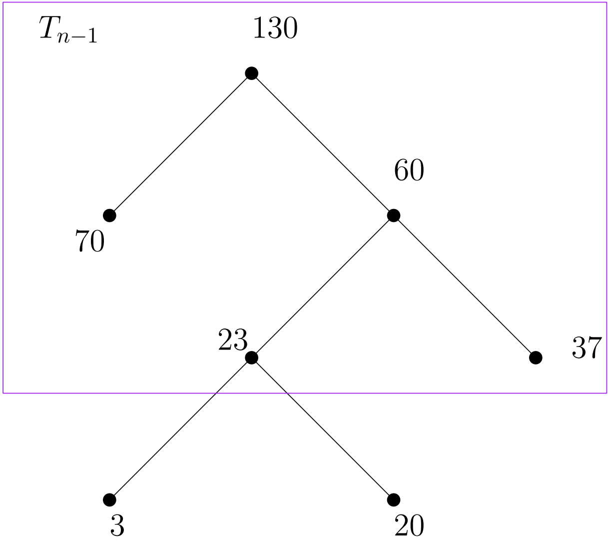
$huffman$ produces an optimal binary prefix code.

induction setup

Let $f_1 \leq f_2 \leq \ldots f_n$ be the original input. From *lightest siblings*, there exists some optimal T_n with f_1 and f_2 as deepest siblings. Let $f_{n+1} = f_1 + f_2$. Let $T_{n-1} = \texttt{huffman}(f_3 \ldots f_{n+1})$.

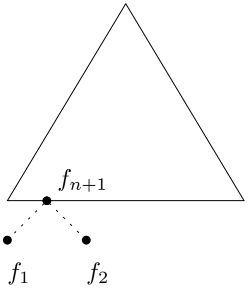


Induction example



Induction details

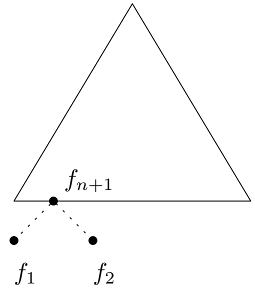
By induction, T_{n-1} is optimal. Now put f_1 and f_2 back as children of f_{n+1}, producing T_n. Let d_i denote the height of f_i in T_{n-1} (and/or T_n)



$$
\begin{aligned}
\operatorname{cost}(T_n) &= \sum_{k=1}^{n} f_k d_k \\
&= f_1 d_1 + f_2 d_2 + \sum_{j=3}^{n+1} f_j d_j - f_{n+1} d_{n+1}
\end{aligned}
$$

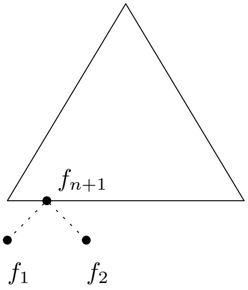
Induction details

$$
\begin{aligned}
\mathrm{cost}(T_n) &= f_1 d_1 + f_2 d_2 + \sum_{k=3}^{n+1} f_k d_k - f_{n+1} d_{n+1} \\
&= \mathrm{cost}(T_{n-1}) + f_1 d_1 + f_2 d_2 - f_{n+1} d_{n+1} \\
&= \mathrm{cost}(T_{n-1}) + (f_1 + f_2) d_1 - f_{n+1}(d_1 - 1) \\
&= \mathrm{cost}(T_{n-1}) + f_1 + f_2 .
\end{aligned}
$$



Induction details

Suppose $\exists$ cheaper T'_n for $f_1 \ldots f_n$. Removing f_1 and f_2 yields a tree T'_{n-1} for $f_3 \ldots f_{n+1}$.



$$\mathrm{cost}(T'_n) = \mathrm{cost}(T'_{n-1}) + f_1 + f_2$$

$$\mathrm{cost}(T'_{n-1}) + (f_1 + f_2) < \mathrm{cost}(T_{n-1}) + (f_1 + f_2)$$

(contradiction)

$$\mathrm{cost}(T'_{n-1}) < \mathrm{cost}(T_{n-1})$$