# CS3383 Unit 5.1: SAT

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# Outline

## Combinatorial Search

- SAT
- Tractable kinds of SAT

# The SAT Problem

## Conjunctive Normal Form (CNF)

Variables $\{ x_1 ... x_n \}$

Literals $L = \{ x_i, \bar{x}_i \mid \text{variable } x_i \}$

Clauses $\{ z_1, ..., z_k \} \subset L$

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**Clauses** $\{ z_1, \ldots, z_k \} \subset L$

## Propositional Satisfiability (SAT)

**Instance** Set of clauses $S$

**Question** $\exists$ setting of variables to $0$, $1$ such that each clause has at least one true literal?

# SAT Example

$$\{\{1,2,3\},\{-1,-2,-3\}\} = \{\{x_1,x_2,x_3\},\{\bar{x}_1,\bar{x}_2,\bar{x}_3\}\}$$

(A) $=$

## Truth Table

| $x_1$ | $x_2$ | $x_3$ | $A$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | |
| | $\vdots$ | | |

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(A) $$= (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

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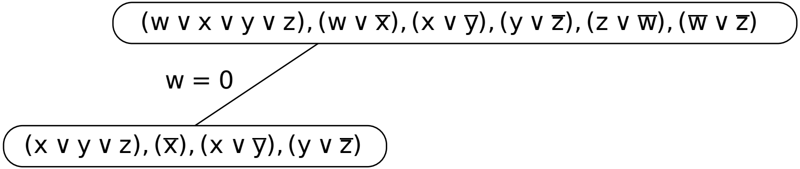
# Backtracking for SAT

**representation** (reduced) clauses

**test** if empty clause, return `False`. If no clauses, return `True`. Otherwise return `None` (UNKNOWN)

**expand** $P_0 = \text{reduce}(P, j, 0)$, $P_1 = \text{reduce}(P, j, 1)$ for some $j$.

# Backtracking SAT Example II



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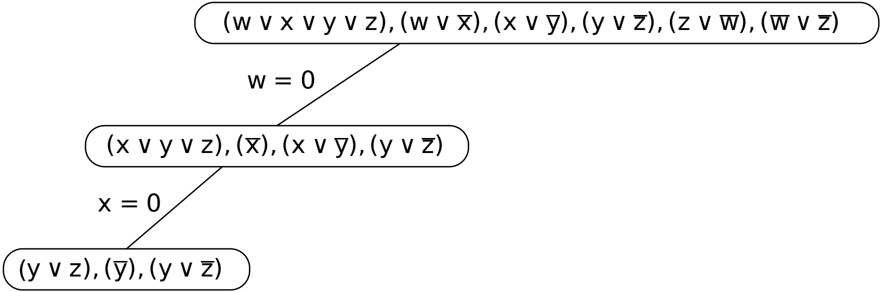

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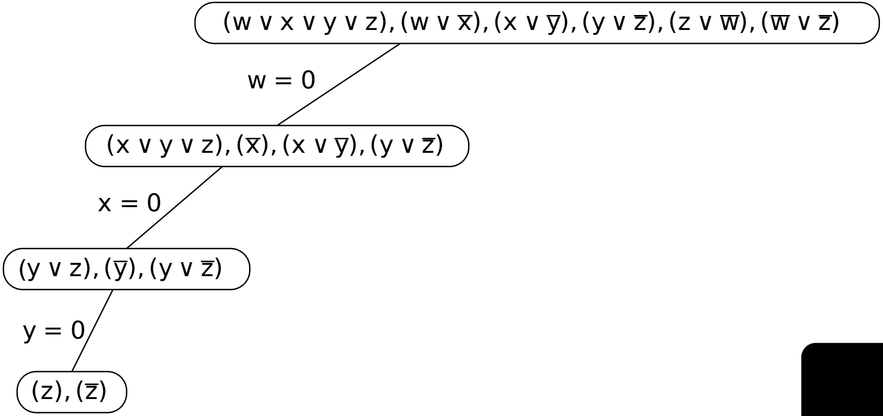

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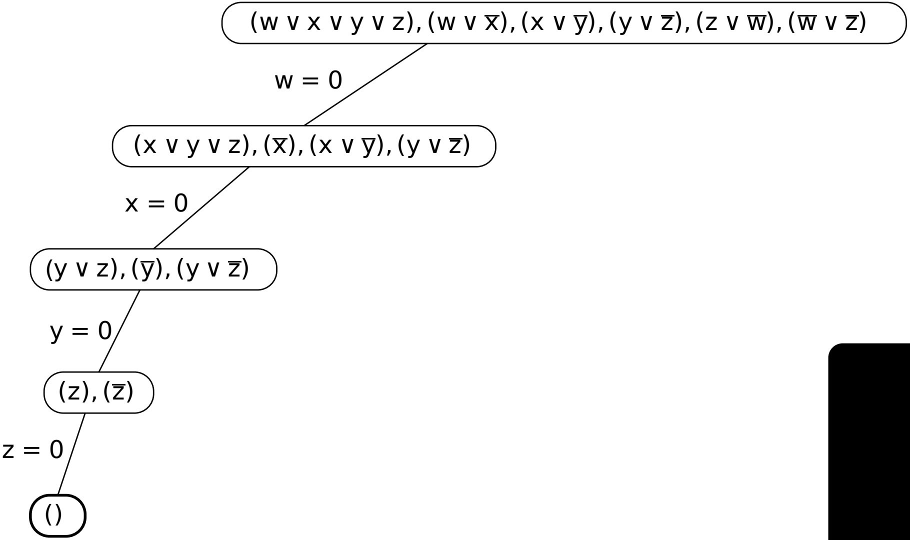

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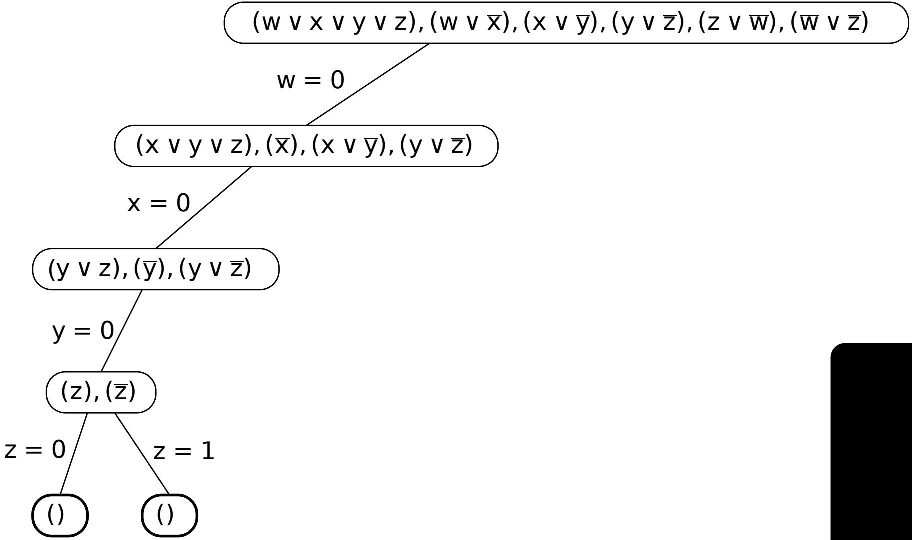

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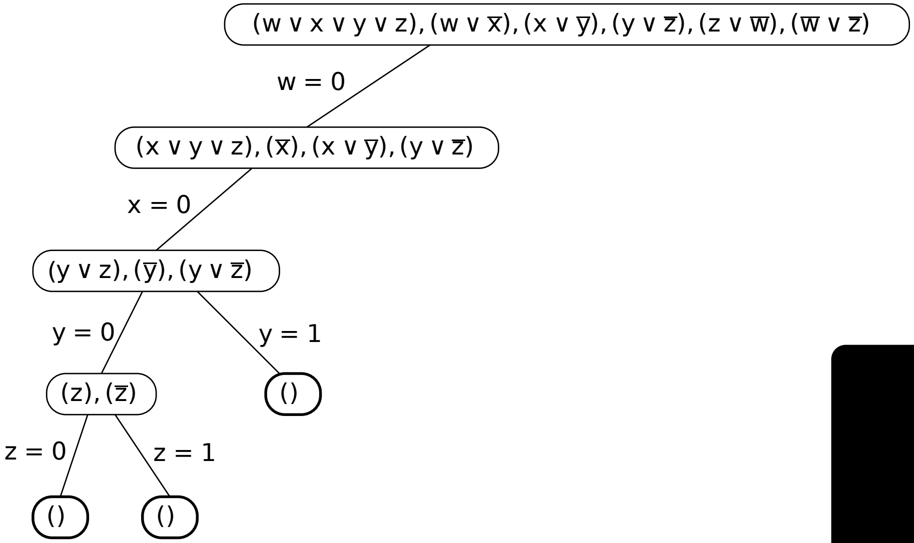

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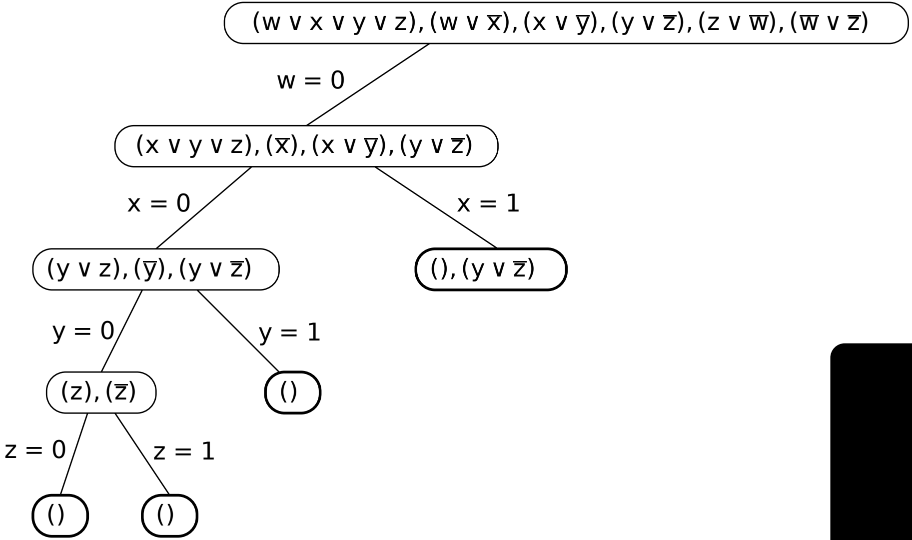

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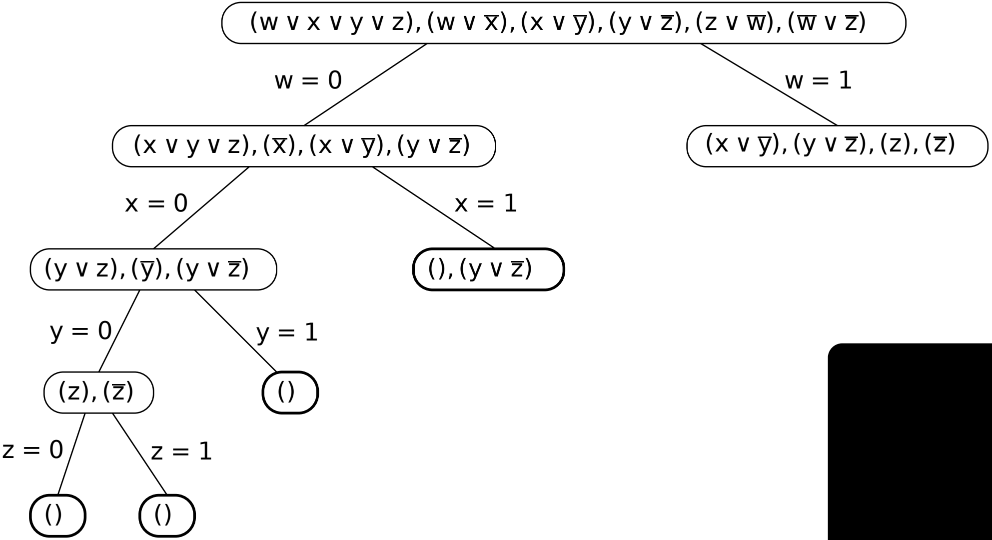

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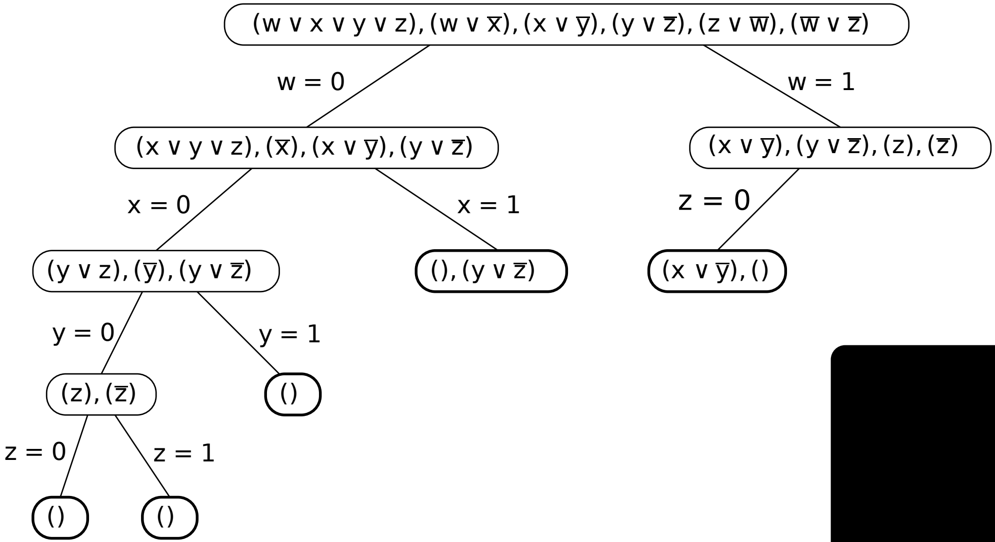

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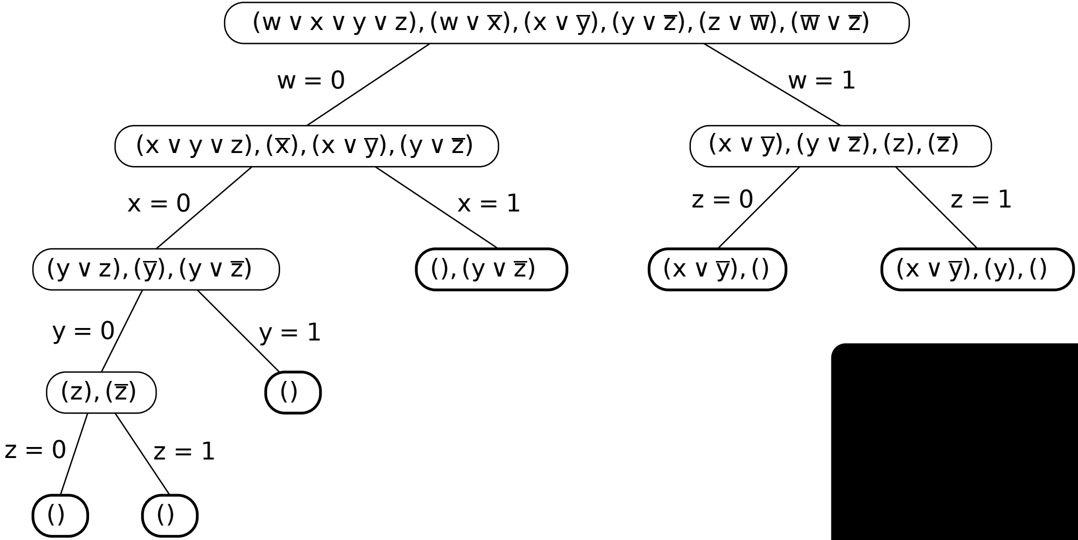


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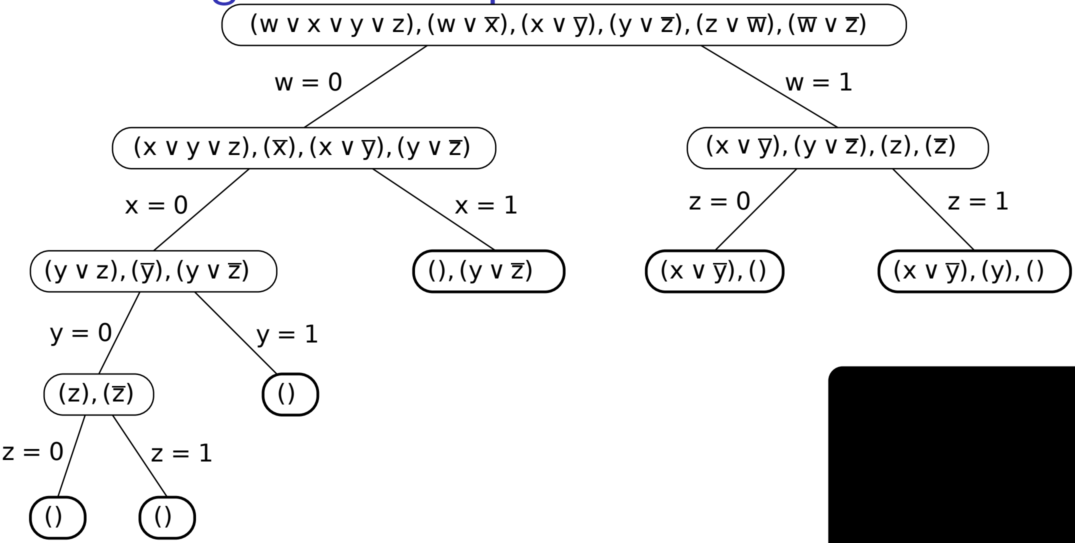


We tried 11 possibilities. Maximum?

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- solvable with *unit propagation*

# Horn SAT

## Horn formulas

implication $(z \wedge w \wedge q) \Rightarrow u$. LHS is all positive, RHS one positive literal

negative clauses $(\bar{x} \vee \bar{w} \vee \bar{y})$. All literals negated.

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- special CNF with at most one positive literal.

# Unit propagation

```
def UnitProp(S):
  Q = [ c for c in S if len(c)==1 ]; V = []
  while len(Q)>0:
    z = Q.pop()[0]; T = []
    V.append(z)
    for C in S:
      C = [j for j in C if j!=-z]
      if len(C)==0: return (False,V)
      if len(C)==1: Q.append(C)
      if not z in C: T.append(C)
    if len(T)==0: return (True,V)
    S = T
  return S
```

# Solving Horn SAT with Unit Propagation

1. Apply unit propagation
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**Proof**

any remaining clause has at least one negative literal