CS3383 Unit 5.1: Travelling Salesperson Problem

David Bremner

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Outline

Combinatorial Search

- Travelling Salesperson problem
- Dynamic Programming for TSP
- Branch and Bound

Travelling Salesperson Problem

TSP

Given $G = (V, E)$

Find a shortest tour that visits all nodes.

Brute Force

- $n!$ different tours

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Brute Force

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- Each one takes $\Theta(n)$ time to test
- Using Stirling's approximation for $n!$

$$n \cdot n! \in \Theta(n^{n+\frac{3}{2}} e^{-n})$$

Subproblems for Dynamic Programming

$C(S, j)$ length of shortest path starting at 1, visiting all nodes in S and ending at j.

Recurrence

$$C(S, j) = \min_{i \in S \setminus \{ j \}} C(S \setminus \{ j \}, i) + d_{ij}$$

Dynamic Programming for TSP

```
C[{1},1] ← 0
for s = 2 to n do
    for ∀ subsets S ∋ 1 of size s do
        C[S,1] ← ∞
        for j ∈ S \ {1} do
            C[S,j] ← min_{i∈S\{j}} C[S \ {j}, i] + d_ij
        end
    end
end
return min_j C[V,j] + d_j1
```

Where:

$C[\{1\},1] \leftarrow 0$

$C[S,1] \leftarrow \infty$

$C[S,j] \leftarrow \min_{i \in S \setminus \{j\}} C[S \setminus \{j\}, i] + d_{ij}$

$\min_j C[V,j] + d_{j1}$

Analysis

- $n \cdot 2^n$ subproblems, each takes linear time: total $\Theta(n^2 2^n)$

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- speedup

$$s(n) \geq \frac{c_1 n^{n+3/2} e^{-n}}{c_2 n^2 2^n}$$
$$\geq c_3 \frac{1}{\sqrt{n}} \left(\frac{n}{2e}\right)^n$$

for $n \geq 6$

$$\geq c_4 2^n$$

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lower bounds

- Suppose we are minimizing some function $f(\cdot)$.
- We need some function lowerbound such that
  - lowerbound$(P_i) \leq f(P_i)$ for all subproblems P_i
  - lowerbound is faster to compute than f

Branch and Bound in General

```
def BranchAndBound(P_0):
    S ← { P_0 }
    best ← ∞
    while S ≠ ∅:
        (P, S) ← pop(S)
        for P_i ∈ expand(P):
            if test(P_i) = SUCCESS:
                best ← min(best, f(P_i))
            elif lowerbound(P_i) < best:
                S ← S ∪ { P_i }
    return best
```

Subproblems for B&B TSP

$[a, S, b]$ path from a to b passing through S

completed by cheapest path from b to a using $V \setminus S$.

P_0 $[a, \{ a \}, a]$

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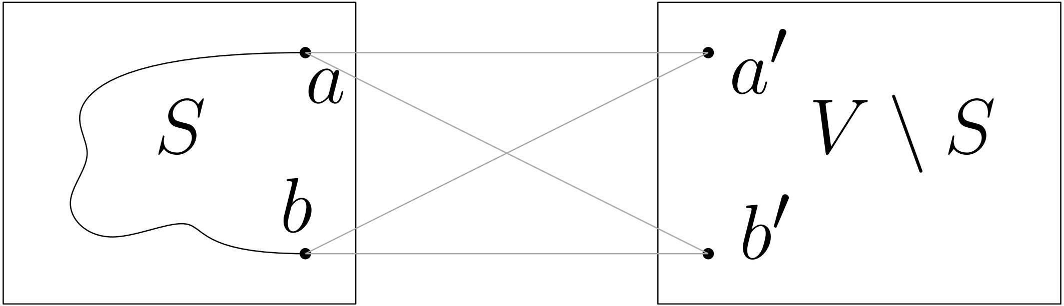
$$\text{expand}([a, S, b]) = \{ [a, S \cup \{ x \}, x] \mid x \in V \setminus S \}$$

Lower bounds from MST



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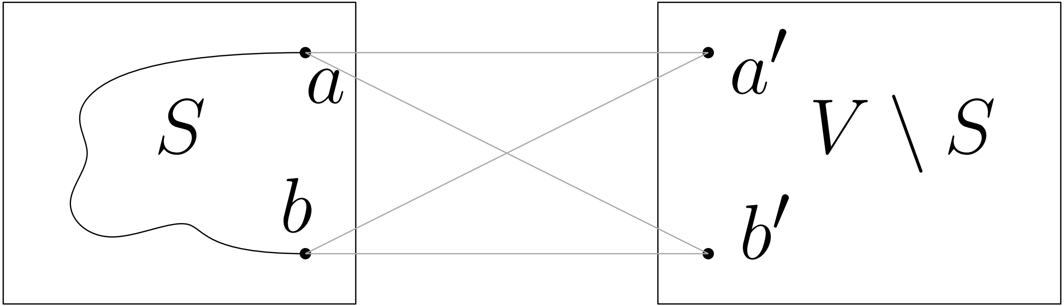

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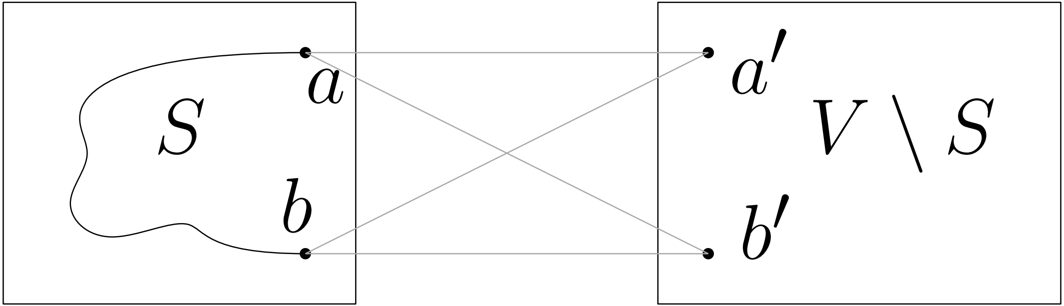

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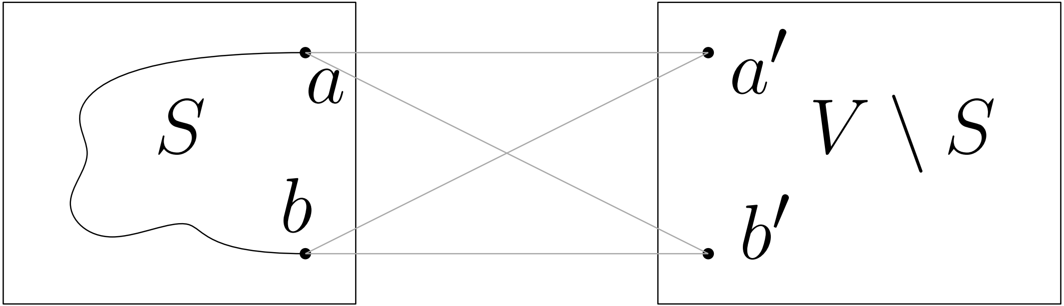

We need to connect

- a to some $a' \in V \setminus S$
- b to some $b' \in V \setminus S$
- a' to b' using all nodes of $V \setminus S$.
- Last is a (special) spanning tree of $V \setminus S$.