CS3383 Lecture 1.3: Quicksort

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# Outline
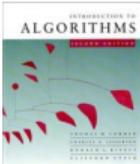
Quicksort demo

```python
def quicksort(A,p,q):
    if p>=q: return

    r = partition(A,p,q,randrange(p,q))
    quicksort(A,p,r-1)
    quicksort(A,r+1,q)
```
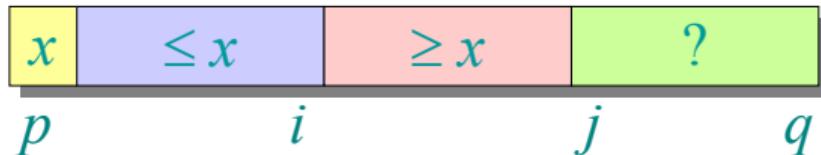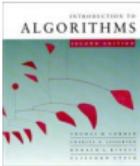
Partitioning subroutine

PARTITION(A, p, q) ▷ $A[p \ldots q]$

    $x \leftarrow A[p]$ ▷ pivot = $A[p]$

    $i \leftarrow p$

    for $j \leftarrow p + 1$ **to** q

        do if $A[j] \leq x$

            then $i \leftarrow i + 1$

                exchange $A[i] \leftrightarrow A[j]$

    exchange $A[p] \leftrightarrow A[i]$

    return i
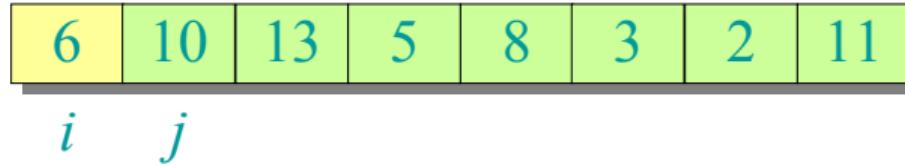
Running time = $O(n)$ for n elements.
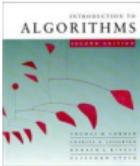
Invariant:
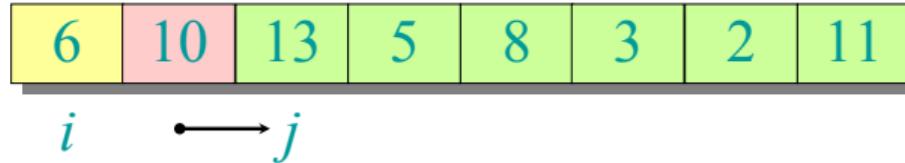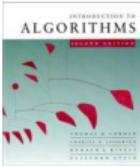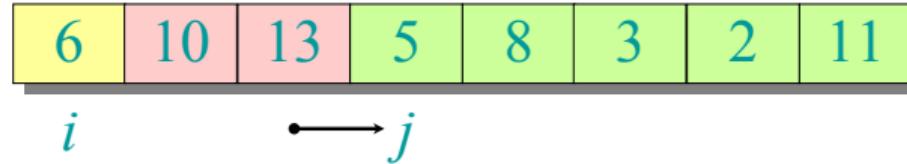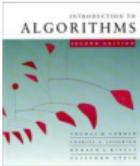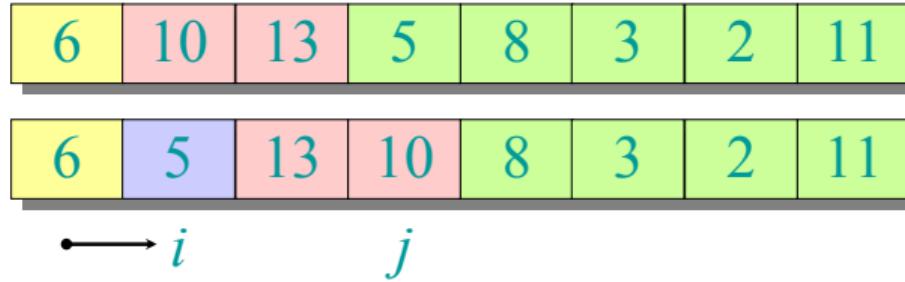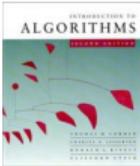
x	$\leq x$	$\geq x$	?
p	i	j	q

Example of partitioning

6	10	13	5	8	3	2	11

i   *j*

Example of partitioning

6	10	13	5	8	3	2	11

i   •⟶ *j*

Example of partitioning

6	10	13	5	8	3	2	11

i j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

$\longrightarrow i \qquad\qquad j$

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

i $\bullet \longrightarrow j$

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Example of partitioning

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6	5	3	10	8	13	2	11

$i \longrightarrow$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

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i j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

i $\longrightarrow$ $\quad j$

Example of partitioning

| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

| 6 | 5 | 13 | 10 | 8 | 3 | 2 | 11 |

| 6 | 5 | 3 | 10 | 8 | 13 | 2 | 11 |

| 6 | 5 | 3 | 2 | 8 | 13 | 10 | 11 |

i $\longrightarrow$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

i j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

2	5	3	6	8	13	10	11

i
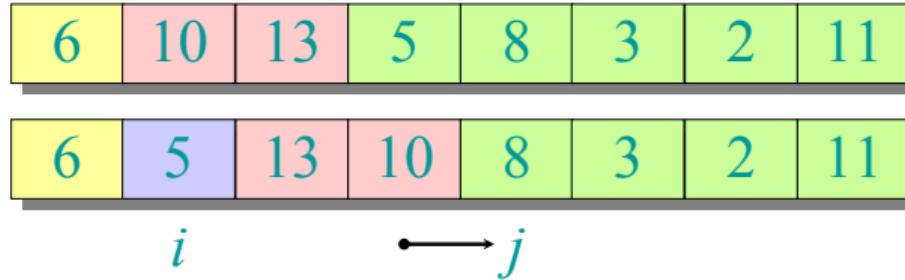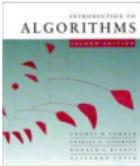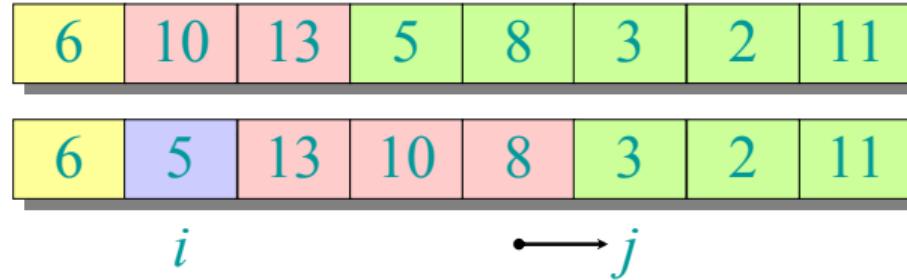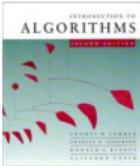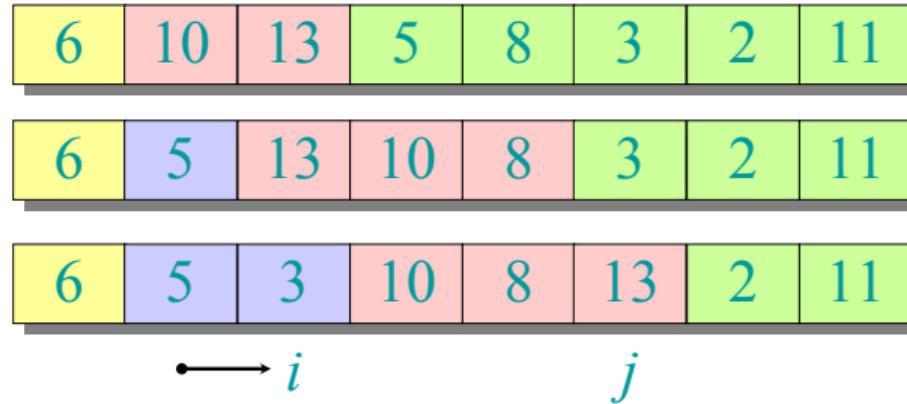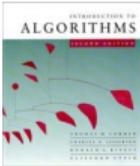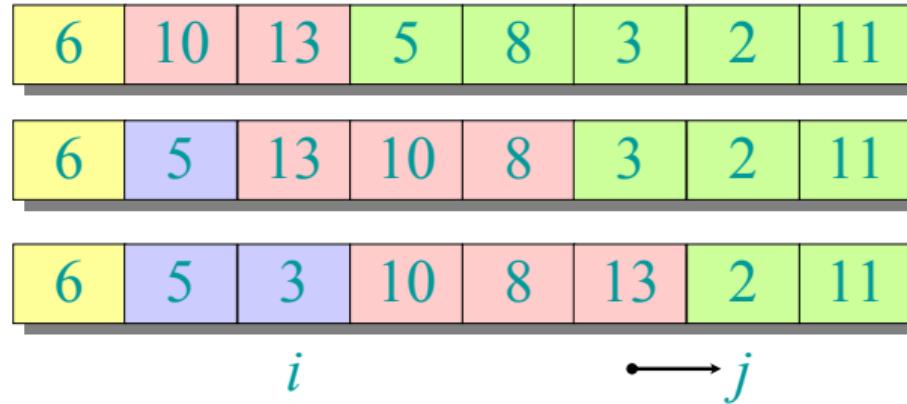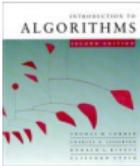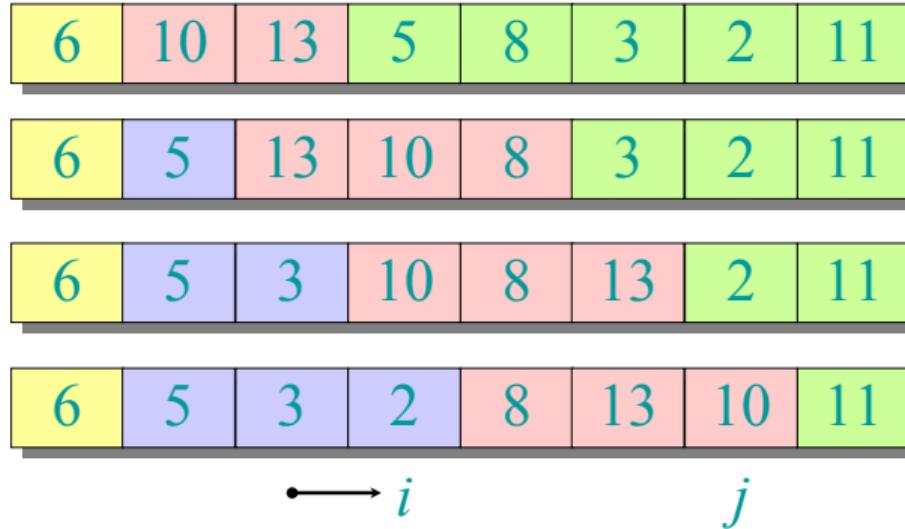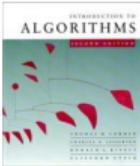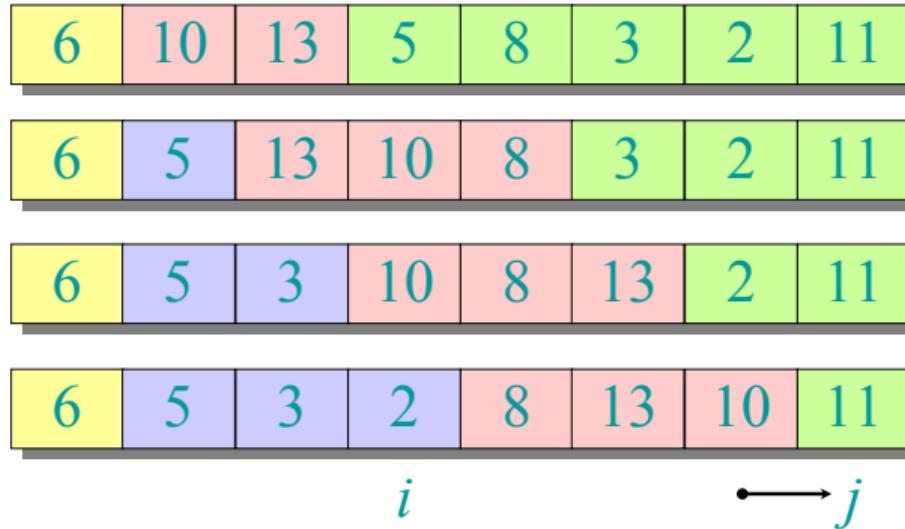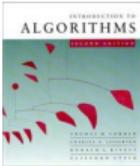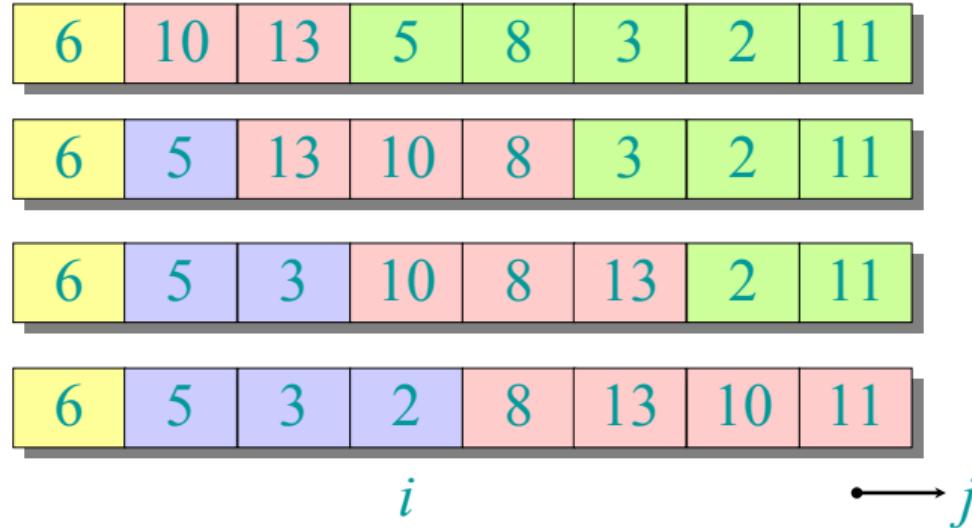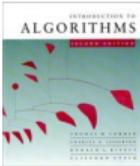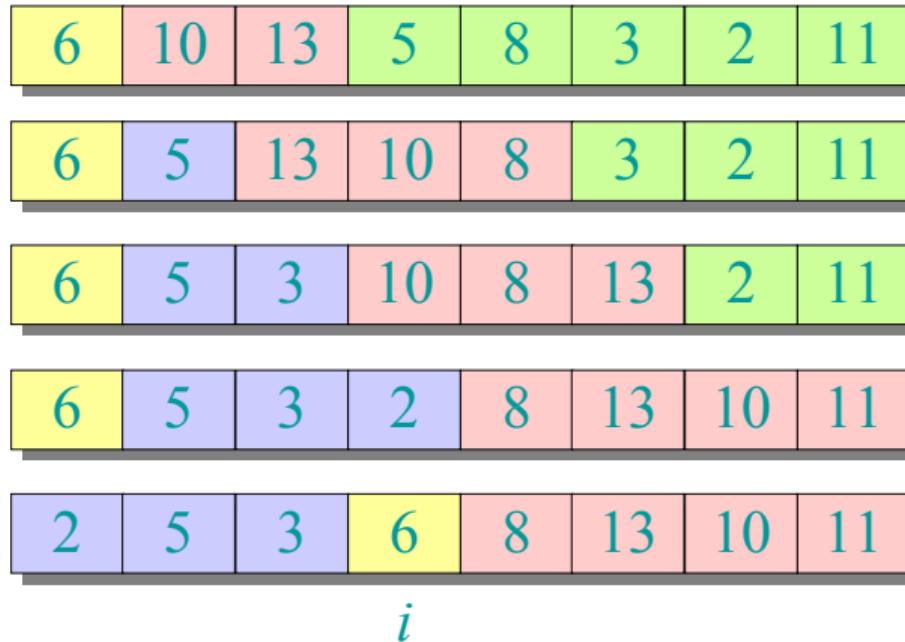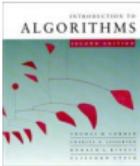
# Random pivot

▶ `pivot ←A[random(1..n)]`
▶ indicator $X_k = 1$ if we generate a $k : n - k - 1$ split, 0 otherwise
▶ $E[X_k] = Pr[X_k = 1] = 1/n$, assuming distinct elements.
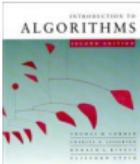
Random pivot demo

```python
def partition(A,p,q,loc):
    A[p],A[loc] = A[loc],A[p]
    i = p
    for j in range(p+1,q+1):
        if A[j] <= A[p]:
            i += 1
            A[i], A[j] = A[j], A[i]
    A[i], A[p] = A[p], A[i]
    return i
```

Analysis (continued)
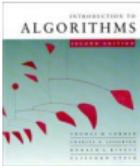
$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n) \big)$$

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$
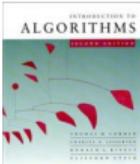
Take expectations of both sides.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$
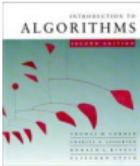
$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

Linearity of expectation.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$
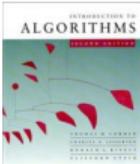
$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\big[T(k) + T(n-k-1) + \Theta(n)\big]$$

Independence of X_k from other random choices.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\big[T(k) + T(n-k-1) + \Theta(n)\big]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$
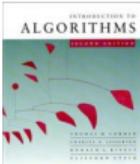
Linearity of expectation; $E[X_k] = 1/n$.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\big[T(k) + T(n-k-1) + \Theta(n)\big]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n) \qquad \text{Summations have identical terms.}$$

Hairy recurrence
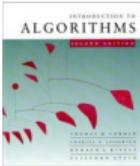
$$E[T(n)] = \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \leq a\,n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a\,n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \geq 2$.
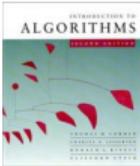
Use fact: $\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).

Substitution method
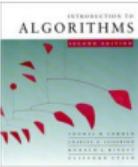
$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$
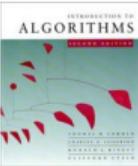
$$\leq \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

Use fact.

Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\leq \frac{2a}{n} \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + \Theta(n)$$

$$= an \lg n - \left(\frac{an}{4} - \Theta(n) \right)$$

Express as *desired – residual*.

Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$= \frac{2a}{n} \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + \Theta(n)$$

$$= an \lg n - \left(\frac{an}{4} - \Theta(n) \right)$$

$$\le an \lg n,$$

if *a* is chosen large enough so that
an/4 dominates the $\Theta(n)$.