Logic Functions with Various Properties

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Outline

Self-Dual Functions

Definition

The dual of a function $f(x_1, x_2, \ldots, x_n)$ is $\overline{f}(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$, denoted by f^d. f^d is obtained first by replacing each x_i with $\overline{x}_i$ and then by complementing the function.

Example

$$\begin{aligned} f &= xy \vee zw \\ f^d &= \overline{(\overline{x} \cdot \overline{y}) \vee (\overline{z} \cdot \overline{w})} \\ &= (x \vee y)(z \vee w) \end{aligned}$$

Definition

A self dual function is a function such that $f = f^d$.

How many self-dual functions?

xyz	$f(x,y,z)$	$f(\overline{x},\overline{y},\overline{z})$	$\overline{f}(\overline{x},\overline{y},\overline{z})$
000	f_0	f_7	$\overline{f}_7$
001	f_1	f_6	$\overline{f}_6$
010	f_2	f_5	$\overline{f}_5$
011	f_3	f_4	$\overline{f}_4$
100	f_4	f_3	$\overline{f}_3$
101	f_5	f_2	$\overline{f}_2$
110	f_6	f_1	$\overline{f}_1$
111	f_7	f_0	$\overline{f}_0$

How many self-dual functions?

Theorem

There are $2^{2^{n-1}}$ different self-dual functions of n variables.

Theorem

Let f be a self-dual function of n variables, and let $|f|$ be the number of inputs a for which $f(a) = 1$, then $|f| = 2^{n-1}$

Theorem

A function which is obtained by assigning a self-dual function to a variable of a self-dual function is also a self-dual function

Definition

A self-anti-dual function is a function such that $f(x_1, x_2, \ldots, x_n) = f(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$

Monotone Increasing Functions

Definition

Let a and b be Boolean vectors. If f satisfies $f(a) \leq f(b)$, for any vectors such that $a \leq b$, then f is a **monotone increasing function** or a **positive function**

Theorem

f is monotone increasing function iff f is a constant or represented by an SOP without complemented literals.

Proof (Monotone increasing $\Rightarrow$ positive SOP)

Let f be a monotone increasing function. Then, we have $f(1, x_2, x_3, \ldots, x_n) \geq f(0, x_2, x_3, \ldots, x_n)$. From this we have

$$f(1, x_2, \ldots, x_n) \vee f(0, x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n) \qquad (1)$$

Shannon's expansion:

$$f(x_1, x_2, \ldots, x_n) = x_1 f(1, x_2, \ldots, x_n) \vee \overline{x_1} f(0, x_2, \ldots, x_n)$$

By applying (1) we have

$$\begin{aligned} f(X) &= x_1(f(1, x_2, \ldots, x_n) \vee f(0, x_2, \ldots, x_n)) \vee \overline{x_1} f(0, x_2, \ldots, x_n) \\ &= x_1 f(1, x_2, \ldots, x_n) \vee (x_1 \vee \overline{x_1}) f(0, x_2, \ldots, x_n) \\ &= x_1 f(1, x_2, \ldots, x_n) \vee f(0, x_2, \ldots, x_n) \end{aligned}$$

Note: x_1 only appears in positive form. (Apply to other vars ...)

Proof (positive SOP $\Rightarrow$ Monotone increasing)

Since f is a positive SOP, we can write:
$f(x_1, x_2, \ldots, x_n) = x_i g_1 \vee g_2$ Let

$$\begin{aligned} c &= (x_1, x_2, \ldots, 0, \ldots, x_n) \\ d &= (x_1, x_2, \ldots, 1, \ldots, x_n) \end{aligned}$$

We have $f(c) = g_2$ and $f(d) = g_1 \vee g_2$. Thus $f(c) \leq f(d)$ holds. Let $a, b \in B^n$ such that $a \leq b$. Consider the sequence of binary vectors such that $a \leq a_1 \leq a_2 \leq \cdots \leq b$ and in $a, a_1, a_2, \ldots, b$, the number of 1's in the vector increases one by one. Then, we have $f(a) \leq f(a_1) \leq f(a_2) \leq \cdots \leq f(b)$. Hence, f is a monotone increasing function.

Monotone Increasing Functions

- The monotone increasing functions with two variables are: $0, x, y, xy, x \vee y$, and 1
- Enumeration in general is not simple

Theorem

A function that is obtained by assigning a monotone increasing function to an arbitrary variable of a monotone increasing function is also a monotone increasing function.

Proof.

A monotone increasing function f is represented by an SOP without complemented literals. Replace a variable with a monotone function $\Rightarrow$ still positive $\Rightarrow$ monotone increasing. □

Monotone Decreasing Functions

- Let a and b be Boolean vectors. If f satisfies $f(a) \leq f(b)$, for any vectors such that $a \leq b$, then f is a monotone decreasing function or a negative function.

Theorem

f is monotone decreasing function iff f is a constant or represented by an SOP with complemented literals only. A monotone decreasing function is obtained by complementing a monotone increasing function.

Question

Enummerate all monotone decreasing functions with two variables.

Unate Functions

Definition

If a function f is a constant or represented by an SOP using either uncomplemented or complemented literals for each variable, then f is a unate function.

Question

How many 2 input variable functions are unate?

Question

Is the function $f(x, y) = x + \overline{xy}$ unate?

Linear Functions

Definition

If a logic function f is represented as

$$f = a_0 \oplus a_1 x_1 \oplus a_2 x_2 \oplus \cdots \oplus a_n x_n$$

where $a_i = 0$ or 1, then f is a linear function.

Question

given a linear function f. Create g by inverting a variable in f. Is g a linear function?

Question

How many linear function of n variables exist?

Linear Functions

Theorem

The function that is obtained by assigning a linear function to an arbitrary variable of a linear function is also a linear function.

Theorem

A linear function is either a self-dual function or a self-anti-dual function.

Proof.

... an exercise for the reader ... □

Symmetric Functions

Definition

A function is a totally symmetric function if any permutation of the variables in f does not change the function. (also called symmetric function)

Definition

If in a function $f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$ is equal to $f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$, then f is symmetric with respect to x_j and x_i. If any permutation of subset S of the variables does not change the function f, then f is a partially symmetric function.

Symmetric Functions

Definition

The elementary symmetric functions of n variables are:

$$
\begin{aligned}
S_0^n &= \overline{x}_1\overline{x}_2\ldots\overline{x}_n \\
S_1^n &= x_1\overline{x}_2\ldots\overline{x}_n \vee \overline{x}_1 x_2\ldots\overline{x}_n \vee \ldots \vee \overline{x}_1\overline{x}_2\ldots x_n \\
\ldots \\
S_n^n &= x_1 x_2\ldots x_n
\end{aligned}
$$

Symmetric Functions

Theorem

An arbitrary n-variable symmetric function f is uniquely represented by elementary symmetric functions

Theorem

Let f and g be totally symmetric functions of n variables. The $f \vee g$, fg, $f \oplus g$, and $\bar{f}$ are also totally symmetric functions.

Theorem

There are ?? symmetric functions of n variables.

Threshold Functions

Definition

Let $(w_1, w_2, \ldots w_n)$ be an n-tuple of real numbers called weights, and t be a real number called thresholds. A threshold function is a function such that:

$$f(x_1, \ldots, x_n) = 1 \iff \sum_{i=1}^{n} w_i x_i \geq t$$

Theorem

A threshold function is a unate function

A majority function is a threshold function. What are the values for w_i? What is t?

Universal Set of Logic Functions

Definition

Let $F = f_1, f_2, \ldots f_m$ be a set of logic functions. If an arbitrary function is realized by a loop-free combinational network using the logic elements that realize functions $f_i (i = 1, 2, \ldots, m)$, the F is universal.

Question

Give an example of universal function set.

Universal Set of Logic Functions

Definition

A function such that $f(0, 0, \ldots, 0) = 0$ is a 0-preserving function. A function such that $f(1, 1, \ldots, 1) = 1$ is a 1-preserving function.

Theorem

The function that is obtained by assigning a 0-preserving function to an arbitrary variable of a 0-preserving function is also a 0-preserving function.

Theorem

The function that is obtained by assigning a 1-preserving function to an arbitrary variable of a 1-preserving function is also a 1-preserving function.

Universal Set of Logic Functions

Theorem

Let

- M_0 *be the set of 0-preserving functions*
- M_1 *be the set of 1-preserving functions*
- M_2 *be the set self-dual functions*
- M_3 *be the set of monotone increasing functions*
- M_4 *be the set of linear functions*

Then, the set of functions F is universal iff

$$F \not\subseteq M_i (i = 0, 1, 2, 3, 4).$$

Proof of necessity

Proof of necessity.

Let $f = \overline{xy}$ then $f \notin M_i (i = 0, 1, 2, 3, 4)$. Each of the sets M_i is closed under the composition of the function. That is, assigning a function in M_i to a variable of M_i, also produces a function in M_i. So if $F \subseteq M_i$, then the function such that $F \notin M_i$ cannot be realized. Thus F is not universal. □

Universal Set of Logic Functions

Lemma

The complement $\overline{x}$ is realized by any non-monotone increasing function and constants 0 and 1

Proof.

□

Example

$f(x, y, z) = x\overline{y} + z$

Universal Set of Logic Functions

Lemma

The AND and OR functions are realized by any non-linear function, complement, and constants 0 and 1

Proof.

□

Example

$f(x, y, z, w) = 1 \oplus x \oplus y \oplus xyz \oplus xywz$

Universal Set of Logic Functions

Lemma

Constants 0 and 1are realized by a non-self-dual function and the complement.

Proof.

□

Example

$f(x, y) = xy$

Universal Set of Logic Functions

Lemma

If f is non 1-preserving and non 0-preserving, then the complement can be realized from f.

Proof.

□

Universal Set of Logic Functions

Lemma

If f is 1-preserving and non 0-preserving, then the constant 1 can be realized from f.

Proof.

□

Universal Set of Logic Functions

Lemma

If f is 0-preserving and non 1-preserving, then the constant 0 can be realized from f.

Proof.

□

The theorem follwos from the 6 lemmas.

Universal Set of Logic Functions

Example

Let $f_1 = \overline{x} \cdot \overline{y}$, $f_2 = x\overline{y}$, $f_3 = x \vee \overline{y}$, $f_4 = x \oplus y$, $f_5 = 1$, $f_6 = 0$, $f_7 = xy \vee yz \vee zx$, $f_8 = x \oplus y \oplus z$, $f_9 = \overline{x}$, $f_{10} = xy$, and $f_{11} = x$.
Show table with functional properties.
Show minimal universal sets.