Computational approaches to polytope diameter questions

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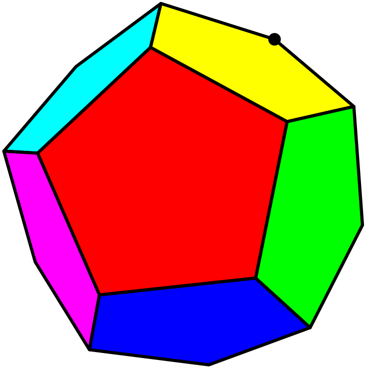
Linear Programming

A linear program

$$\text{maximize} \quad c^T x$$
$$\text{Such that}$$
$$Ax \leqslant b$$

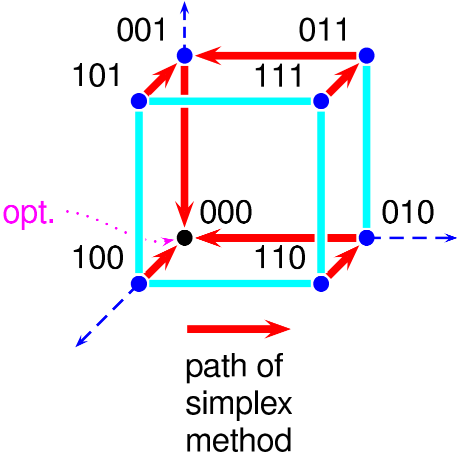
- $P = \{ x \mid Ax \leqslant b \}$ is called a (convex) *polyhedron*
- Bounded polyhedra are called (convex) *polytopes*.

Polytopes



- Face: $\cap$ with supporting hyperplane
- Vertices: faces of dimension 0.
- Edges: faces of dimensions 1

The Simplex Method



Diameter

- $d(u, v) \equiv$ length of the shortest edge-path from u to v.
- diameter $\equiv \max_{(u,v)} d(u, v)$

Hirsch and d-step

Conjecture (Hirsch, 1957)

The maximum diameter $\Delta(d, n)$ of a d-dimensional convex polytope with n facets is at most $n - d$.

Conjecture (Klee and Walkup, 1967)

$\Delta(d, 2d) \leqslant d$

Lemma (Klee and Walkup, 1967)

$\Delta(d, d + k) \leqslant \Delta(k, 2k)$ *with equality for* $k \leqslant d$

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Bounds

Lemma (Klee and Walkup 67, Klee and Kleinschmidt 1987, Kalai 1992)

1. $\Delta(3, n) = \left\lfloor \frac{2}{3} n \right\rfloor - 1$
2. $\Delta(d, 2d + k) \leqslant \Delta(d - 1, 2d + k - 1) + \left\lfloor \frac{k}{2} \right\rfloor + 1$ *for* $0 \leqslant k \leqslant 3$
3. $\Delta(d, n) \leqslant 2(2d)^{\log_2(n)}$

Lemma (Goodey 1972)

1. $\Delta(4, 10) = 5$ *and* $\Delta(5, 11) = 6$
2. $\Delta(6, 13) \leqslant 9$ *and* $\Delta(7, 14) \leqslant 10$

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Table: Bounds on $\Delta(d,n)$ circa 1972.

	$n-d$			
d	4	5	6	7
4	4	5	5	{6,7}
5	4	5	6	[7,9]
6	4	5	{6,7}	[7,9]
7	4	5	{6,7}	[7, 10]

A computational approach

- Consider case with known upper bound $\Delta(n, d) \leqslant k$
- Find all possible combinatorial types of edge paths of length k.
- Show that none of these is realizable as the diameter of an (n, d) polytope.
- It follows $\Delta(n, d) \leqslant k - 1$

Remark

By a perturbation argument, we need only consider the diameter of simple polytopes.

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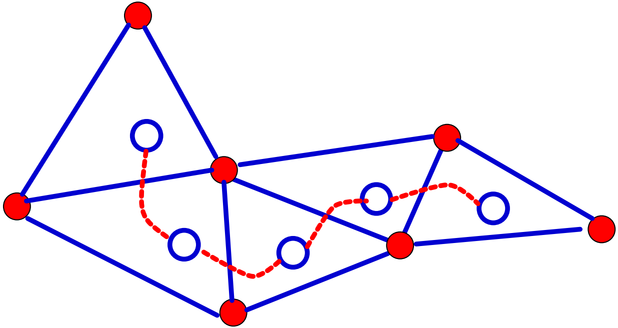
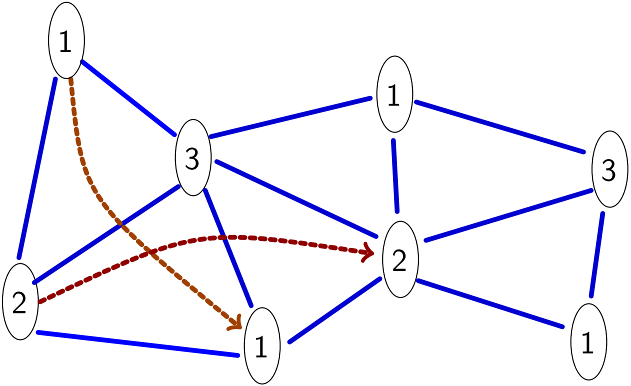
The polar view

- facet paths
  - abstract simplicial complex
  - dual is a path
- pivot sequences



The polar view

- facet paths
- pivot sequences
    - Label initial simplex
    - Label of entering=label of leaving



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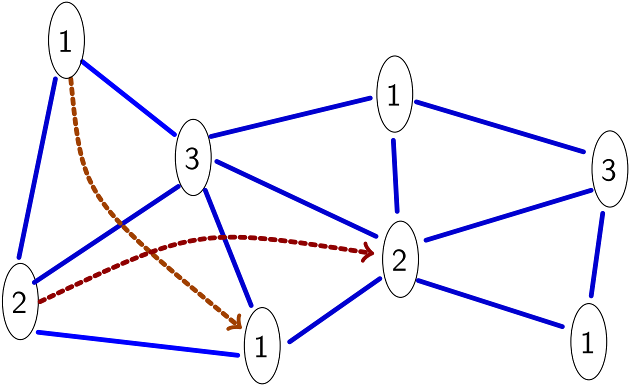
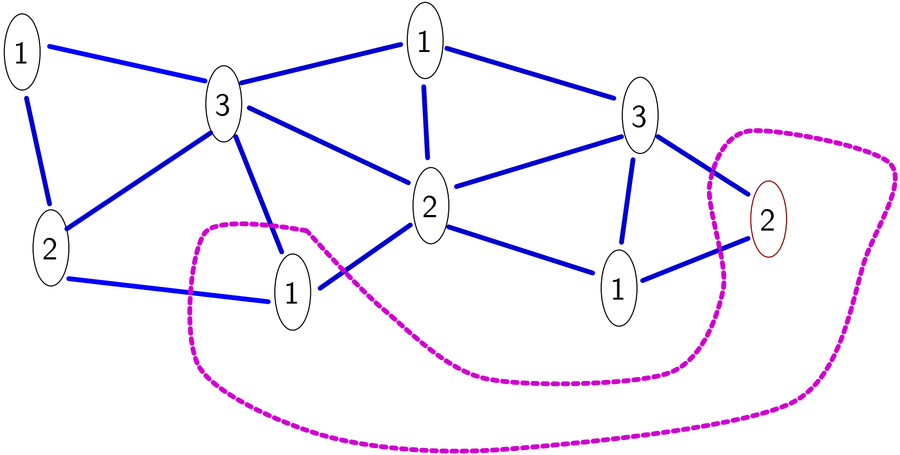


The polar view

- facet paths
- pivot sequences
- labels do not repeat, w.l.o.g., occur in order $\equiv$ *restricted growth strings*, $d-1$ symbols occur in order.

$$
\begin{aligned}
&\mathit{rgs}\ r\ k \mid r > k = [\,]\\
&\mathit{rgs}\ 1\ k = [\mathit{replicate}\ k\ 1]\\
&\mathit{rgs}\ r\ k = \mathit{new\_sym} +\!\!+ \mathit{old\_sym}\\
&\quad \textbf{where}\\
&\qquad \mathit{new\_sym} = [l +\!\!+ [r] \mid l \leftarrow \mathit{rgs}\ (r-1)\ (k-1)];\\
&\qquad \mathit{old\_sym} = [l +\!\!+ [s] \mid l \leftarrow \mathit{rgs}\ r\ (k-1), s \leftarrow [1\,..\,r]];
\end{aligned}
$$

Single revisit paths via identifications



Single revisit paths via identifications

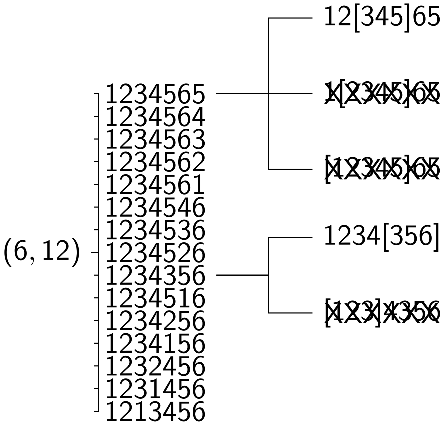
(6, 12)

1234565
1234564
1234563
1234562
1234561
1234546
1234536
1234526
1234356
1234516
1234256
1234156
1232456
1231456
1213456

12[345]65
1[2345]65
[12345]65

1234[356]
[123]4356

Single revisit paths via identifications



Lemma

Every combinatorial type of end-disjoint single revisit path has an encoding as pivot sequence without a revisit on the first facet.

Polytope boundary completion

Problem

Given abstract simplicial complex Δ, is there a simplicial polytope whose boundary complex contains Δ.

- NP Hard (Richter-Gebert)
- Algebraically difficult (arbitrary sets of polynomial inequalities).

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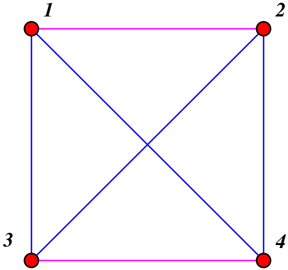
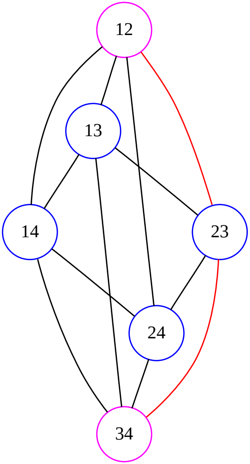
Shortcuts

- *pivot graph*: nodes $\equiv$ (potential) facets, edges $\equiv$ (potential) ridges
- inclusion minimal paths: $\Pi = F_0, F_1, \ldots F_k$, where no subset of Π is a path from F_0 to F_k.
  - can be generated recursively





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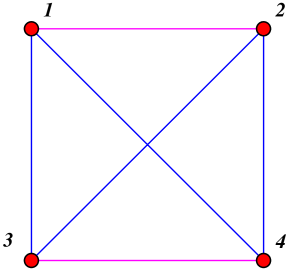

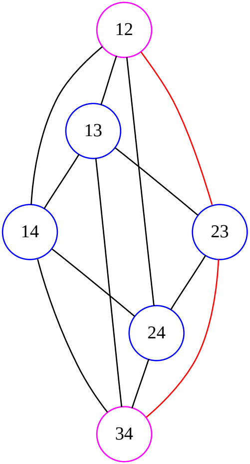

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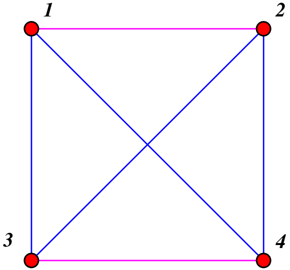

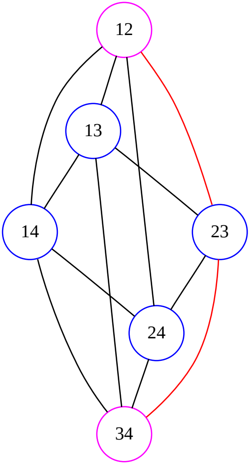

Geodesic Embedding

Problem

Given path complex Γ, *and a set* $\Pi_1 \ldots \Pi_m$ *of forbidden path complexes on the same ground set, is there a simplicial polytope whose boundary complex contains* Γ, *but not any* Π_i.

Remark

For a no answer, it suffices to find a contradiction with some valid set of constraints.

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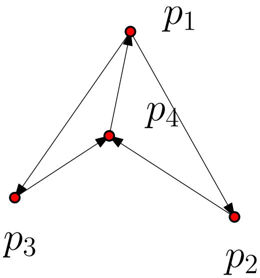
Realizability and Chirotopes

- Given $P = \{(q_i, 1)\} \subset \mathbb{R}^{d+1}$,

$$\chi(i_1, \dots i_{d+1}) = \operatorname{sign} |p_{i_1}, \dots p_{i_{d+1}}|$$

- For any set of points $\chi()$ obeys the *Graßman-Plücker relations*
- We call any alternating map χ obeying the G-P relations a *chirotope*.



$$\chi(1,2,3) = -1$$
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Remark

No chirotope for some constraints means no point set for those constraints.

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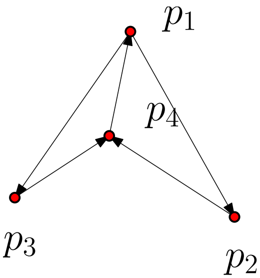

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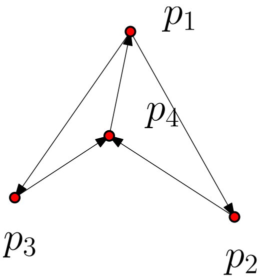

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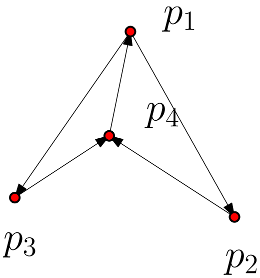

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Chirotopes and SAT

- Uniform case (no zero determinants)
- 3-term Graßmann-Plücker Constraints. For $\lambda \in N^{d-1}$, $a, b, c, d \in N \setminus \lambda$.

$$|P_\lambda\ p_a\ p_b||P_\lambda\ p_c\ p_d| - |P_\lambda\ p_a\ p_c||P_\lambda\ p_b\ p_d| + |P_\lambda\ p_a\ p_d||P_\lambda\ p_b\ p_c| = 0$$

$$\neq \{\chi(\lambda\ a\ b) = \chi(\lambda\ c\ d), \chi(\lambda\ a\ c) \neq \chi(\lambda\ b\ d), \chi(\lambda\ a\ d) = \chi(\lambda\ b\ c)\}$$

yields $16\binom{n}{d-1}\binom{n-d+1}{4}$ CNF constraints.

- Facet constraints can be dealt with in preprocessing.
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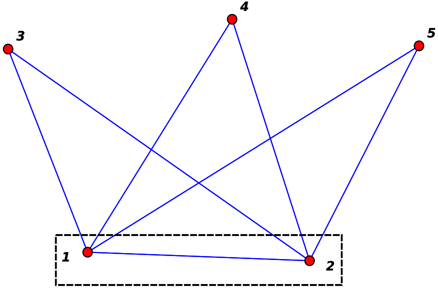

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$$\neq \{\, \chi(F_1 a_1), \chi(F_1 b_1) \dots \sigma_{12}\chi(F_2 a_2), \sigma_{12}\chi(F_2 b_2) \,\}$$

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Computational Results

- For $(6,12)$, 10 cases, each taking a few hours on a laptop.
- For $(4,11)$, 35 cases, each taking at most a few hours.
- For $(5,12)$, 540 cases, 19 taking more than 48 hours.

Table: Summary of bounds for $\Delta(d,n)$. The bold entries are from the computations discussed in this talk.

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